Machine Dynamics

Chapter 5b : Mechanical Vibration

Department of
Mechanical Engineering
National Taiwan University

Tseng-Ti Fu
Room 503-5
Email: ttfu@ntu.edu.tw

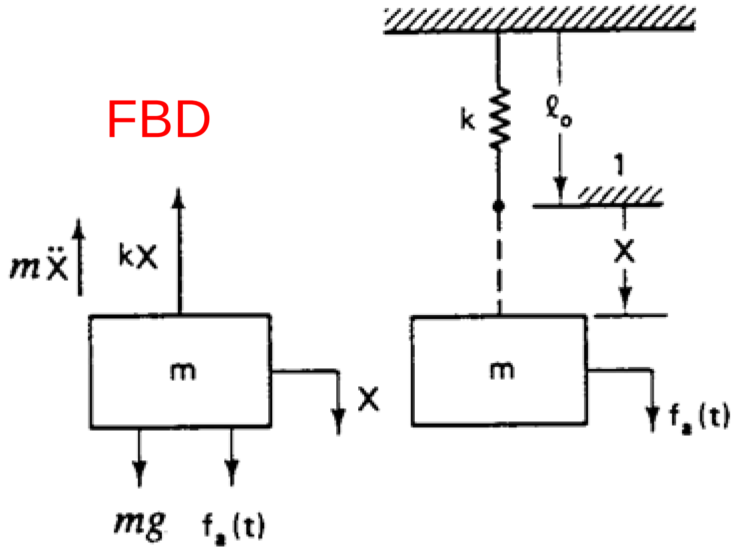
工程評估實驗室

Translational vibration

- Vibration mass without damping

1) By D'Alembert principle

$$m\ddot{X} + kX = mg + f_a(t)$$



2) By Lagrange equation

a) For X is measured from the state of the spring and is used as the

$$V = \qquad\qquad\qquad , \quad T = \frac{1}{2}m\dot{X}^2, \quad L = T - V$$

Translational vibration (cont.)

The Lagrange equation of motion is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right)-\frac{\partial L}{\partial X}=Q_X$$

b) For x is measured from the state of the mass.

$X = l + x$ where (equilibrium)

$V =$

$T = \frac{1}{2}m\dot{x}^2, \quad \dot{X} = \dot{x}$

Translational vibration (cont.)

The Lagrange equation of motion becomes:

$$m\ddot{x} + kx + kl - mg = f_a(t)$$

Note:

① 同時適用於 的振動系統。

② There is no need to consider

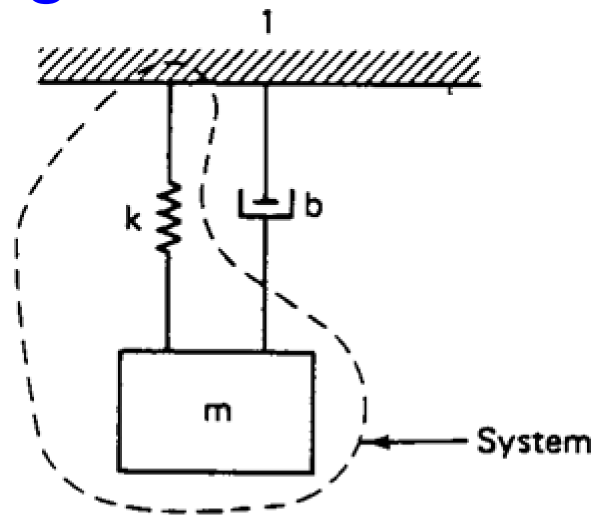
③ It leads to a EOM only when the terms involved are all

④ The potential energy can also take the following simple form:

Translational vibration (cont.)

- Vibration mass with damping

For b is the damping coefficient and the damping force is proportional to the



The Lagrange equation of motion is:

Translational vibration (cont.)

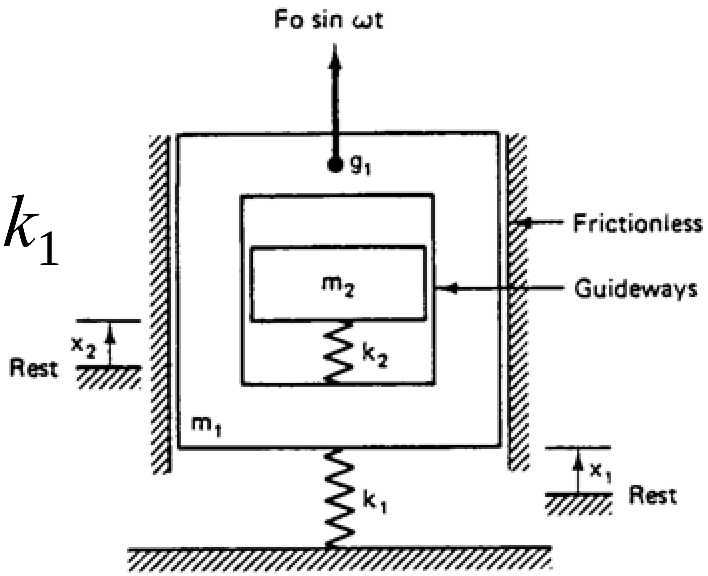
- Two mass vibration without damping

1) The responses of a two-mass system without damping to a force:

The system can represent a simplified 1-D model of a

a) machine m_1 : rigid & supported by k_1

b) machine m_2 : fastened by k_2 on the frame m_1



Translational vibration (cont.)

Since x_1 & x_2 are defined from the equilibrium positions of m_1 & m_2

$$T = \frac{1}{2} m_1 \dot{x}_1^2$$

$$V = \frac{1}{2} k_1 x_1^2$$

※ Remove the potential energy of gravity due to the definitions of x_1 & x_2

$$Q_{x1} =$$
$$Q_{x2} =$$

Translational vibration (cont.)

The Lagrange equations of motion are:

$$\begin{cases} m_1\ddot{x}_1 + \\ m_2\ddot{x}_2 + \end{cases}$$

※The transient responses will decay in a very short time so that the operation is the dominant mode.

The steady state solutions of x_1 & x_2 are obtained by assuming:

Translational vibration (cont.)

Insert them into the EOM, yields

$$\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]X_1-$$

$$-X_1+\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]X_2=$$

where

$X_0=\qquad,\quad \omega_1=\qquad\qquad,\quad \omega_2=$

Translational vibration (cont.)

※ $\omega_1 \& \omega_2$ are the natural frequencies of the mass-spring systems when each is mounted on a stationary frame,

Solutions of the amplitudes are:

$$\frac{X_1}{X_0} = \frac{}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}}$$

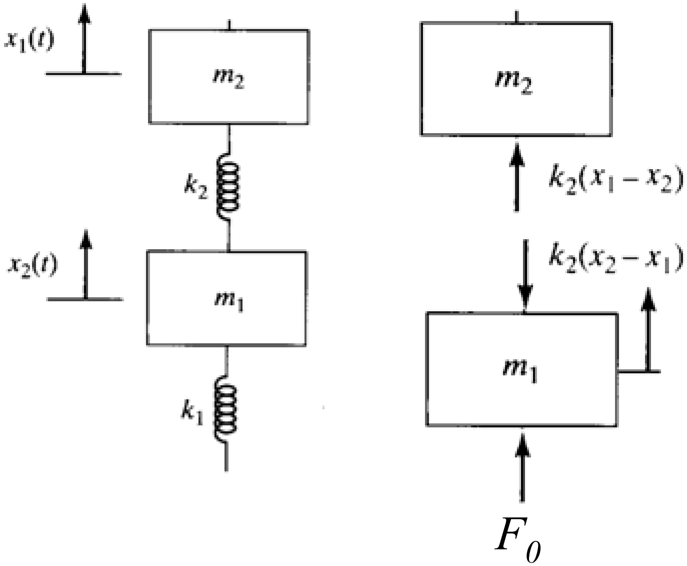
$$\frac{X_2}{X_0} = \frac{}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}}$$

Translational vibration (cont.)

2) The “absorbing” effect →

$$X_1 = \quad ,$$

$$X_2 = -\frac{k_1}{k_2} X_0 =$$



※ Physical meaning:

$$= \frac{k_2 X_2 \sin \omega_2 t}{F_0 \sin \omega_2 t} = -1$$

Translational vibration (cont.)

∴The displacement of m_2 is
with the applied force so does the transmitted spring force. The applied force is " " by the moving mass , leaving no force to excite the frame m_1.

→ acts as a vibration absorber to remove the troublesome
of a machine with the expense of introducing

→ reduce vibration by
rather than consuming

Translational vibration (cont.)

3) The natural frequencies of the system:

By setting the   of the steady state amplitudes   , the result is the   equations and its solutions yield the system natural frequencies:

$$\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]$$

$$\text{let} \quad r=\frac{m_2}{m_1}, \quad \therefore \frac{k_2}{k_1}=$$

Translational vibration (cont.)

if

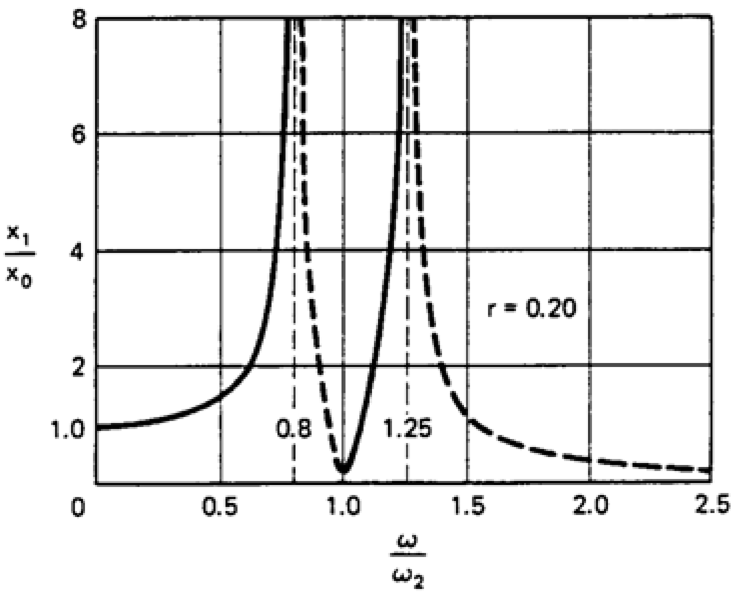
The above equation is simplified to

For $r << 1$,

$$\left(\frac{\omega}{\omega_2}\right) \approx$$

$\therefore$ If $r = 0.2$,

$\omega =$



Translational vibration (cont.)

※ Characteristics of the system:

a) As , operation range

b) Most effective when the operation frequency is

c) Rapidly pass through the natural frequency & avoid overrunning into the one.

d) Each of the resonance is the result of a particular mass making its contribution through one of its

Translational vibration (cont.)

e) The characteristic equation is a system and of how the system being excited.

→ can also obtained from the differential equations

Assume the solutions to the homogeneous equations are:

$$\left.\begin{aligned} x_1 &= A_1 \sin(\omega t + \phi_1) \\ x_2 &= A_2 \sin(\omega t + \phi_2) \end{aligned}\right\}$$

regardless of the $f_a(t)$ form

Translational vibration (cont.)

$$\left\{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\right.$$
$$\left.-A_1+\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]\right.$$

A common ratio of can only be obtained for specific values of , which are the natural frequencies of the system.

Translational vibration (cont.)

$$\frac{A_1}{A_2} = \frac{\dfrac{k_2}{k_1}}{1 + \dfrac{k_2}{k_1} - \left(\dfrac{\omega}{\omega_1}\right)^2} =$$

※ Can also be treated geometrically and in a matrix form so that:

$$\frac{A_1}{A_2}:$$

The natural frequencies :

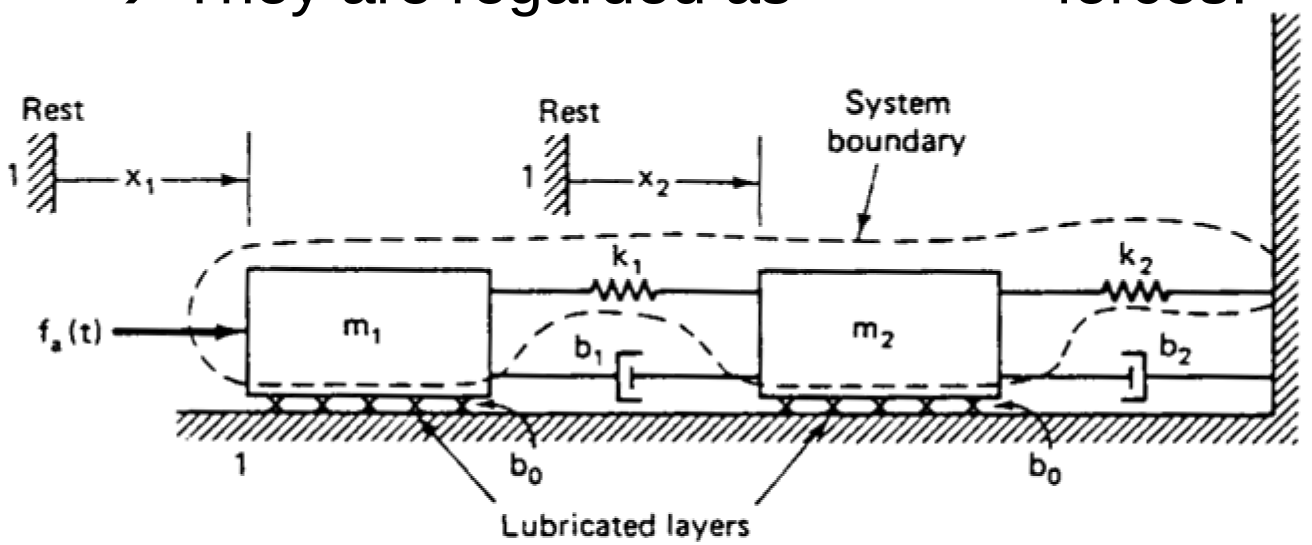
Translational vibration (cont.)

- Two mass vibration with damping

  1) Assumptions：

     a) The system excludes the & mass-supporting layers.

     → They are regarded as forces.



Translational vibration (cont.)

b) Masses move

→ Fluid films can be treated as linear viscous damping as follows:

shear stress $\tau = \mu \dfrac{du}{dy} =$

$$F = \tau \cdot A \propto v \;\rightarrow\; F =$$

c) x_1, x_2 are measured from the positions when

.

2) Lagrange terms

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad V =$$

Translational vibration (cont.)

G. C.:

$$Q_{x1} =$$
$$Q_{x2} =$$

3) Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=Q_{x_1}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=Q_{x_2}$$

$$m_1\ddot{x}_1+b_1\left(\dot{x}_1-\dot{x}_2\right)+$$
$$m_2\ddot{x}_2+b_1\left(\dot{x}_2-\dot{x}_1\right)+$$

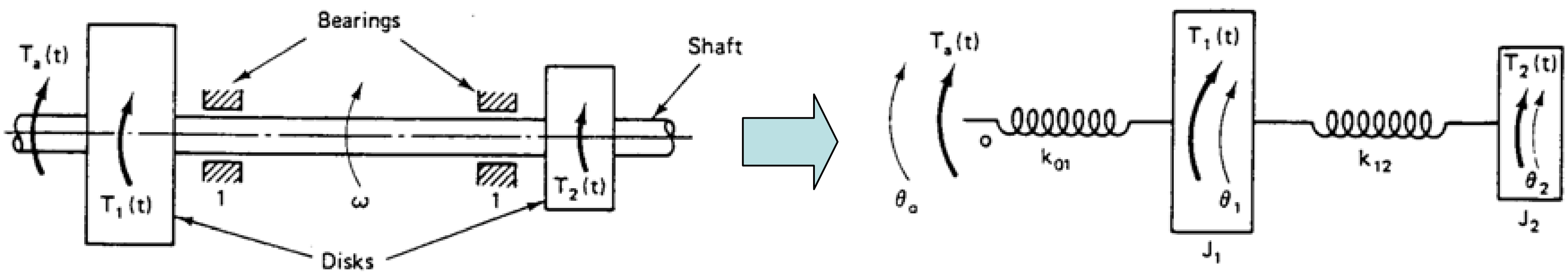
Torsional vibration

- Basic system elements

1) disk：used to store

2) shaft：used to store

- Physical configuration vs. mass model



Torsional vibration (cont.)

- Lagrange analysis

K.E.: $T =$

P.E.: $V =$

If $\theta_0, \theta_1, \theta_2$ are measured from rest.

G.C.:

The Lagrange EOMs:

a) For θ_0, $Q_{\theta 0} =$

Torsional vibration (cont.)

b) For θ_1, $Q_{\theta_1} =$

c) For θ_2, $Q_{\theta_2} =$

※ The number of EOMs can be reduced by substituting a) into b) & obtain:

$$I_1\ddot{\theta}_1 + k_{12}(\theta_1 - \theta_2) =$$

$$I_2\ddot{\theta}_2 + k_{12}(\theta_2 - \theta_1) =$$

Torsional vibration (cont.)

1) For the case of

The above equations can be decoupled by adding and subtracting processes

$$(\mathrm{a})+(\mathrm{b}) \quad I\left(\ddot{\theta}_1+\ddot{\theta}_2\right)=$$

$$(\mathrm{a})-(\mathrm{b}) \quad I\left(\ddot{\theta}_1-\ddot{\theta}_2\right)+$$

$$\text{Let} \quad \begin{aligned} \phi_1 &= \theta_1+\theta_2 \\ \phi_2 &= \theta_1-\theta_2 \end{aligned}$$

(Decoupled equations)

Torsional vibration (cont.)

From the homogeneous solutions to see its natural frequencies

$$\phi_1 = c_{11}$$

$$\phi_2 = c_{21}\sin\sqrt{\frac{2k_{12}}{I}}t$$

$\therefore$ Two natural frequencies are:

a) For the initial conditions of

（初始角度、初始角速度相同）

Torsional vibration (cont.)

$\phi_1(0) = c_{11} = \theta_1(0) + \theta_2(0) = \text{constant}$

$\phi_2(0) = c_{22} = \theta_1(0) - \theta_2(0) = 0$

$\phi_1 =$

$\phi_2 =$

$\therefore \quad \theta_1(t) = \theta_2(t) =$

→ motion occurs, both disks rotate at the speed.

→ no energy is stored in & the whole system rotates

Torsional vibration (cont.)

b) For initial conditions of:

（初始角度相同、初始角速度不相同）

$$\phi_1 = [\dot{\theta}_1(0) + \dot{\theta}_2(0)]t$$

$$\phi_2 = [\frac{\dot{\theta}_1(0) + \dot{\theta}_2(0)}{\quad}]$$

Torsional vibration (cont.)

$$\theta_1 = \frac{1}{2}\left[\dot{\theta}_1(0)+\dot{\theta}_2(0)\right]t + \left[\frac{\dot{\theta}_1(0)+\dot{\theta}_2(0)}{\sqrt{\frac{2k_{12}}{I}}}\right]\sin\sqrt{\frac{2k_{12}}{I}}t$$

$$\theta_2 = \frac{1}{2}\left[\dot{\theta}_1(0)+\dot{\theta}_2(0)\right]t$$

→ The whole system rotates at a constant angular velocity while the disks oscillate about the mean angle:

Torsional vibration (cont.)

$\therefore$ rigid body mode $f_n =$

oscillation mode $f_n =$

2) For the case of

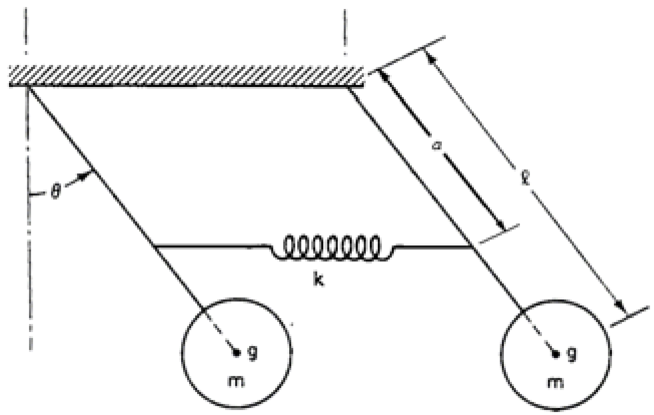
→ rigid body mode $f_n =$

oscillation mode $f_n =$

Torsional vibration (cont.)

- Examples of other systems with rigid body mode

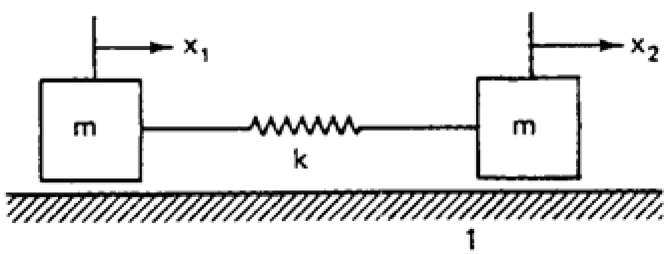
1)



$\omega \Rightarrow$ $\dfrac{\theta_1}{\theta_2} = 1$

$\Rightarrow$ $\dfrac{\theta_1}{\theta_2} = -1$

2)

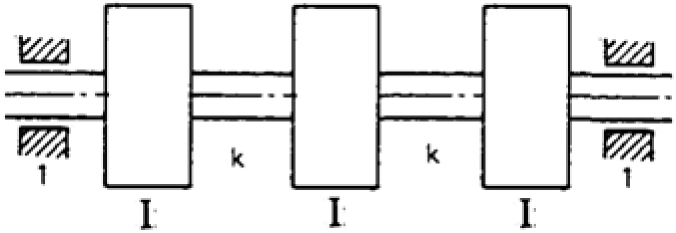


$\omega \Rightarrow$ $\dfrac{x_1}{x_2} = 1$

$\Rightarrow$ $\dfrac{x_1}{x_2} = -1$

Torsional vibration (cont.)

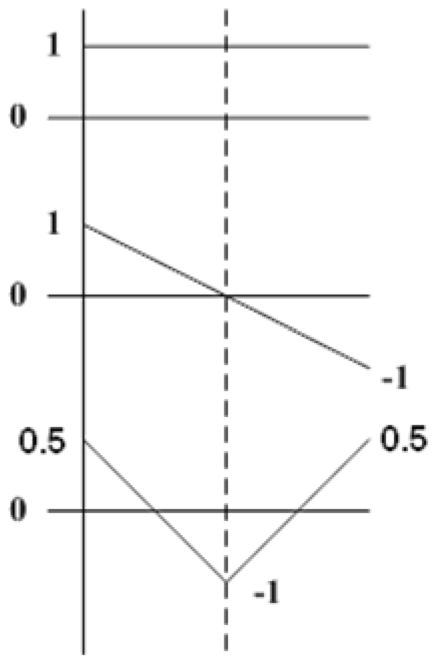
3)



$\omega \Rightarrow$

$\Rightarrow$

$\Rightarrow$



$x_1 : x_2 : x_3 \Rightarrow$

$\Rightarrow$

$\Rightarrow$

The mode shapes

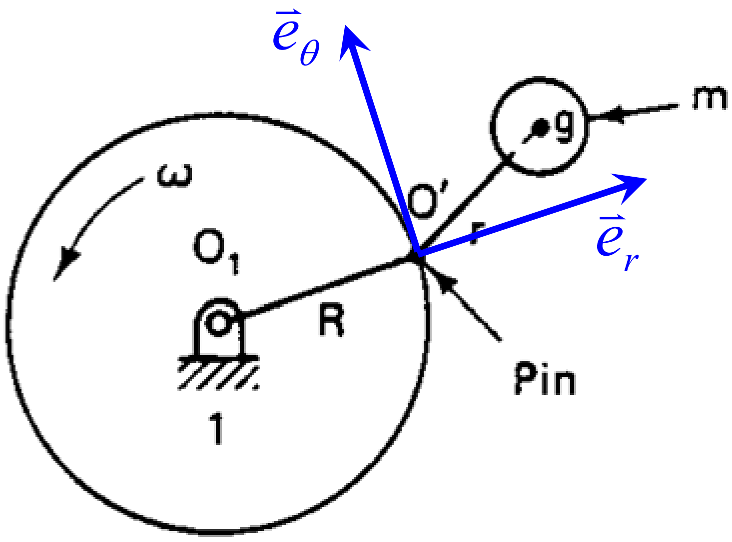
Torsional vibration (cont.)

- Vibration absorber

  1) An absorber whose "f_n" changes with the system's   so that it is effective over a   of operation.

  2) Lagrange approach

     The velocity of m is established by the orthogonal coordinates   &   with O' as the origin.



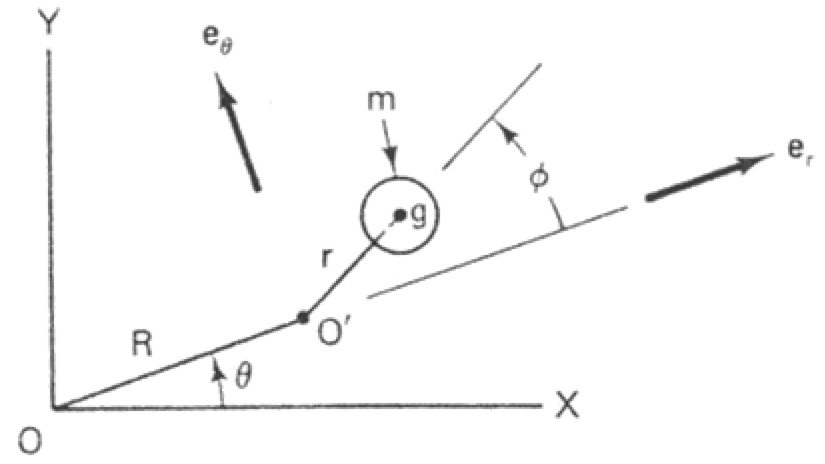
Torsional vibration (cont.)

$$V_g = V_{O'} + V_{g/O'}$$
$$= R\dot{\theta}\vec{e}_\theta + \vec{\omega}\times\vec{r}$$
$$= R\dot{\theta}\vec{e}_\theta + r\left(\dot{\theta}+\dot{\phi}\right)\cos\phi\,\vec{e}_\theta -$$

$$\vec{\omega} = \left(\dot{\theta}+\dot{\phi}\right)\vec{k}$$

K.E. ： $T = \frac{1}{2}m\left\{\left[R\dot{\theta} + r\left(\dot{\theta}+\dot{\phi}\right)\cos\phi\right]^2\right.$

P.E. ： neglected



Torsional vibration (cont.)

G.C. → (∵ is the input.)

$= 0$

∴The EOM：

※ Linearise the EOM by

a) let

b) assume

$$\ddot{\phi} + \left(\frac{R}{r} \dot{\theta}^2 \right) \phi =$$

Excitation from the speed change of the system

Torsional vibration (cont.)

$\omega_n =$, $\dot{\theta}$ is the shaft rotation speed

→ If the shaft rotates at speed n with a superimposed oscillation $\theta_0 \sin \omega t$, where is the frequency of oscillation.

The shaft angular displacement is:

$$\theta = nt + \theta_0 \sin \omega t$$

velocity

acceleration

where θ_0 & n are constant.

Torsional vibration (cont.)

→ If the oscillation is small, i.e.

$$\therefore \dot{\theta} \approx n$$

$$\ddot{\phi} + \left(\frac{R}{r} n^2 \right) \phi =$$

Assume $\phi = \phi_0 \sin \omega t$

$$\frac{\theta_0}{\phi_0} =$$

$$\therefore \text{If } \omega = \qquad \rightarrow \theta_0 = 0$$

Torsional vibration (cont.)

3) Example：

For a six-cylinder 4-stroke reciprocating engine crankshaft to be used with a pendulum absorber.

$\omega =$

$$\frac{\theta_0}{\phi_0} = \frac{\frac{R}{r}n^2 - 9n^2}{\left(\frac{R+r}{r}\right)9n^2} = \underline{\qquad\qquad}$$

∴If the absorber is designed by inspective of shaft operation speed n.

Torsional vibration (cont.)

→ 設計條件

1)

2)

3)

→ The natural frequency of the pendulum:

∵ The ω_n tracks the frequency.

∴ The absorber is at engine speeds.

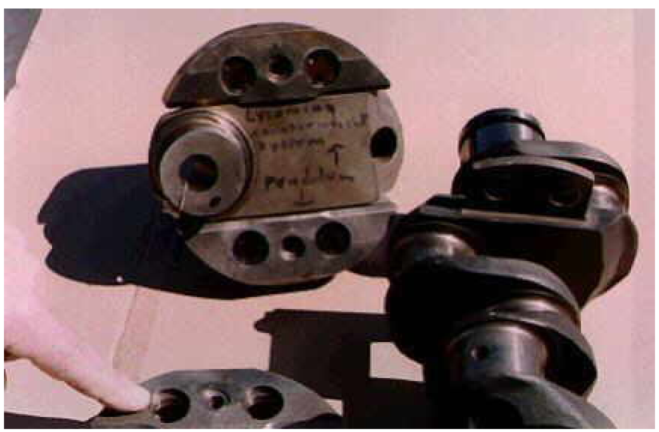
Torsional vibration (cont.)

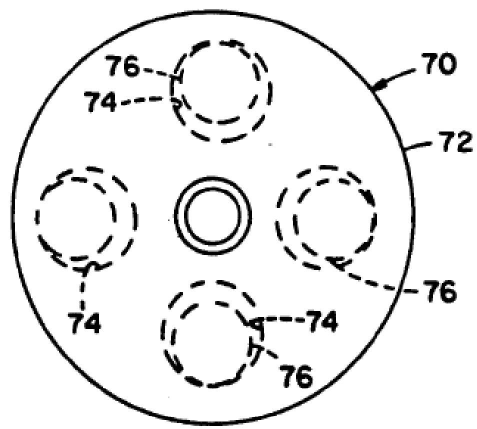
4) Examples of applications

   a) Automotive engines

      → Application in the shaft of an internal combustion engine (US5295411)

   a) Helicopter rotors





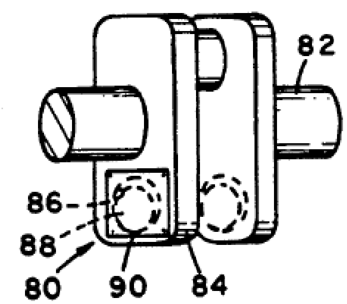


Fig. 9

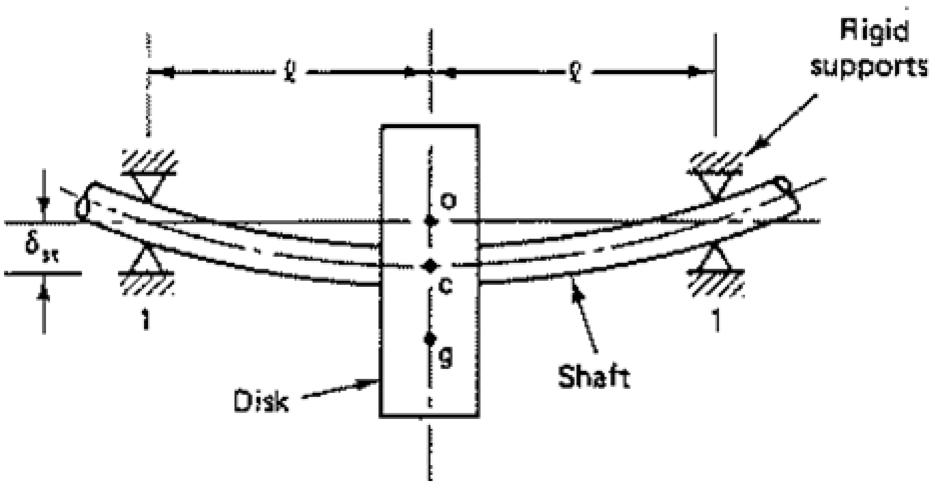
The Whirling Effect

- A rotor mounted on a support

1) The shaft and rotor resembles a with concentrated or distributed loads

：the C.G. of shaft + disk

：the geometric center of the rotor.



The Whirling Effect (cont.)

: $\overline{cg}$, eccentricity

: gravity-induced sag

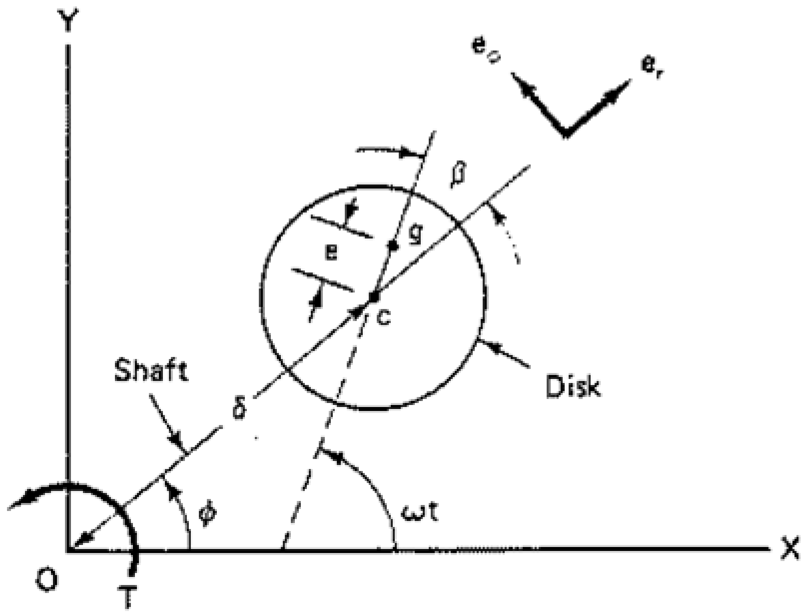
: the stiffness of the shaft

ω : the angular velocity of the system

β : the phase angle

: the angular displacement of the "bow plane"

$\dot{\phi}$: the speed



The Whirling Effect (cont.)

Use the polar coordinates:

$$\dot{\vec{\delta}} =$$

$$\vec{r}_{cg} = e\cos\beta\vec{e}_r + e\sin\beta\vec{e}_\phi$$

$$\dot{\vec{r}}_g = \dot{\vec{\delta}} + \vec{\omega}\times\vec{r}_{cg}$$

$$= (\dot{\delta} - e\omega\sin\beta)\vec{e}_r + (\delta\dot{\phi} + e\omega\cos\beta)\vec{e}_\phi$$

$$K.E.:$$

Neglect the gravity

$$P.E.:$$

The Whirling Effect (cont.)

G.C.：

$$\because dW = T \cdot d\omega = T(d\phi + d\beta)$$

The EOMs are：

$$\delta : \frac{d}{dt}\left[m(\dot{\delta} - e\omega \sin\beta)\right]$$

$$\phi : \frac{d}{dt}\left\{m\left[\delta^2\dot{\phi} - e\dot{\delta}\sin\beta + \omega\left(e^2 + k_r^2\right) + e\delta\left(\omega + \dot{\phi}\right)\cos\beta\right]\right\}$$
$$= T$$

The Whirling Effect (cont.)

$$\beta : \frac{d}{dt}\left\{ me\left[\omega e\left(1+\frac{I_g}{me^2}\right) - \dot{\delta}\sin\beta + \delta\dot{\phi}\cos\beta \right]\right\}$$

2) Simplification

a) For machines running at a speed

(=constant)

$\dot{\omega} = 0$, $\ddot{\beta} = -\ddot{\phi}$ ($\because$)

EOMs: $\ddot{\delta} +$ $= e\omega^2 \cos\beta \cdots\cdots (1)$

$\delta\ddot{\phi} + 2\dot{\delta}\dot{\phi} =$ $\cdots\cdots\cdots\cdots (2)$

The Whirling Effect (cont.)

In which, is the natural frequency of the non-rotating rotor (i.e. the speed).

b) Assume =constant

→

→ called "synchronous precession" or

" "

from (2), $2\dot{\delta}\omega = e\omega^2 \sin\beta \cdots\cdots (3)$

$\Rightarrow$

The Whirling Effect (cont.)

from (1), $\delta = \dfrac{e\omega^2}{\omega_n^2 - \omega^2}\cos\beta$

$\Rightarrow$

from (3), $e\omega^2 \sin\beta = 0$

$\Rightarrow$

$\therefore$ from $\beta =$ const., it leads to

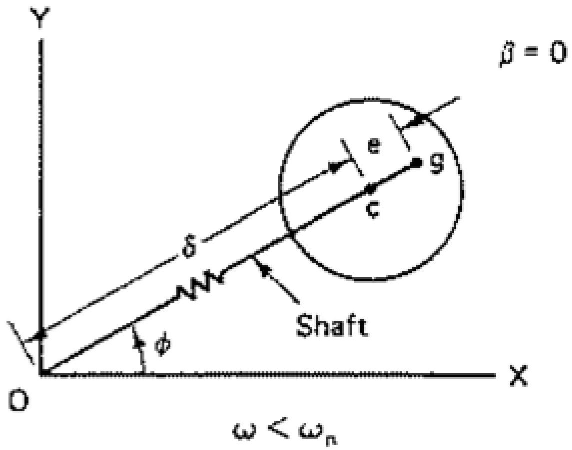
①

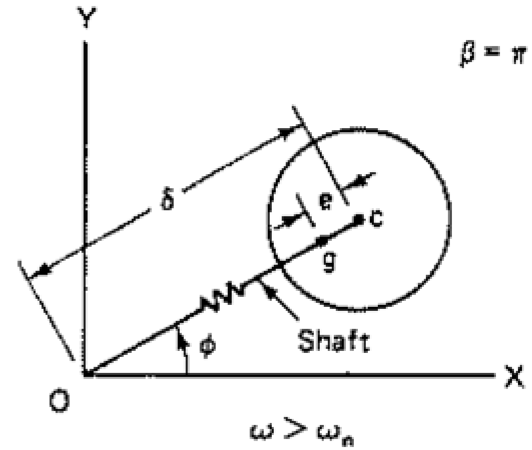
②

③

The Whirling Effect (cont.)

∴ For For





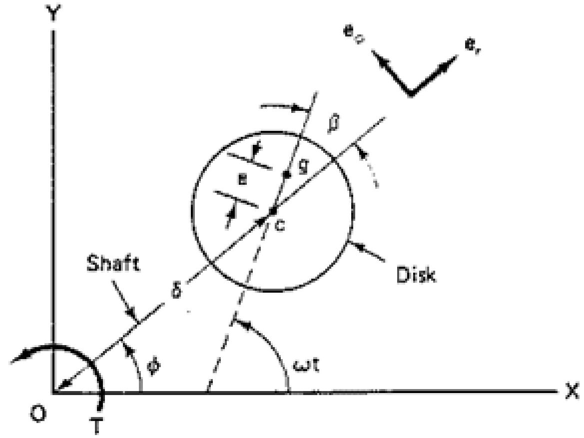
※Note：

The above results are only "
end point" behavior of the system. No account for transient state like

The Whirling Effect (cont.)

3) Considering the damping effect
Assume $\omega =$ const. & there is a viscous damping force acting at C.

$$a_g = a_c + a_{g/c}$$
$$= \left[\left(\ddot{\delta} - \delta\dot{\phi}^2\right) - e\dot{\beta}^2 \cos\beta\right]\vec{e}_r$$
$$+$$



By the d'Alembert principle, the EOMs in the directions of $\vec{e}_\theta$ & $\vec{e}_r$ are:

The Whirling Effect (cont.)

$$\begin{cases} \ddot{\delta} + \dfrac{c}{m}\dot{\delta} + \qquad\qquad = e\omega^2 \cos\beta \\ \delta\ddot{\phi} + \qquad\qquad = e\omega^2 \sin\beta \end{cases}$$

For the steady state responses of synchronous whirl：

$$\dot{\phi} = \omega , \qquad \beta = \text{const.},$$

$$\tan\beta = \frac{\quad}{\left(\dfrac{k_r}{m} - \omega^2\right)} \;, \quad \delta = \frac{\quad}{\sqrt{\left(k_r - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

The Whirling Effect (cont.)

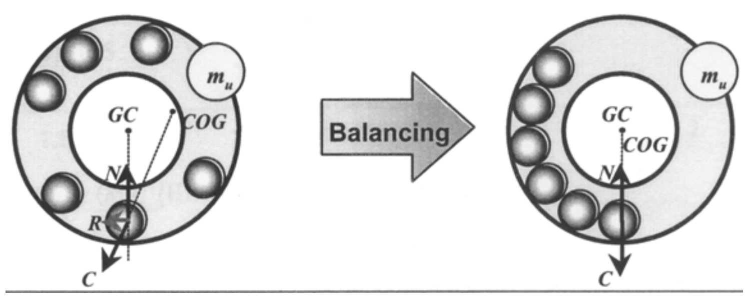
※The responses are the same as the case 2 in basic vibration types.

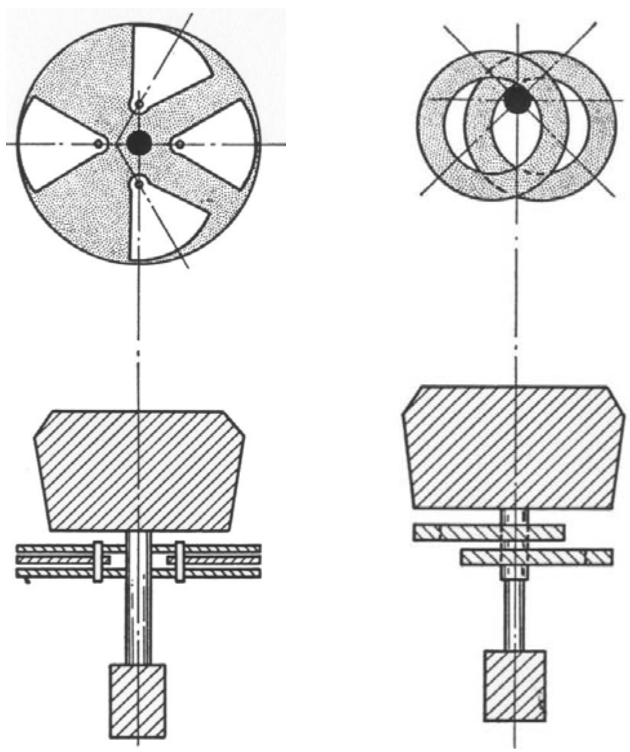
4) Applications：

→ Laundry machines

→ Automatic dynamic balancers

→ Hand held tools





END of Chap_5