

Machine Dynamics

Chapter 4b : Lagrange Equations



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工程評估實驗室



Examples

- (1) Rolling cylinder with/without slip
- (2) Single pendulum
- (3) Travelling inverted pendulum
- (4) Rotating pendulum
- (5) The gyroscopic effect



Rolling cylinder

- Rolling without slip

∴ rolling

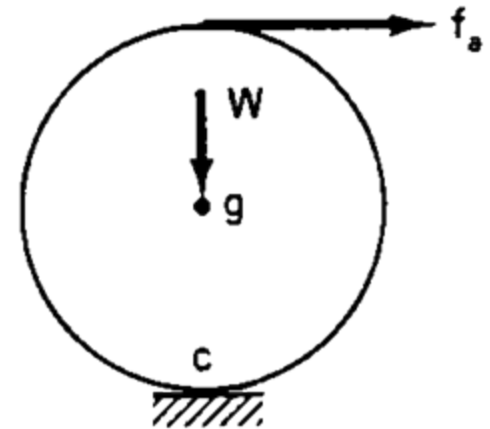
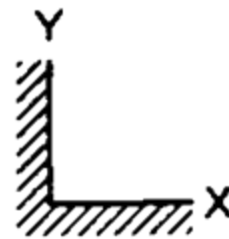
→ no slip at the

→ the and force do no work

→

∴ Required independent generalized coordinate

→





Rolling cylinder (cont.)

The kinetic energy is:

$$T = \frac{1}{2} m \dot{R}_g^2 + \frac{1}{2} I_{zz} \dot{\theta}^2$$

=

or by

=

Lagrangian:



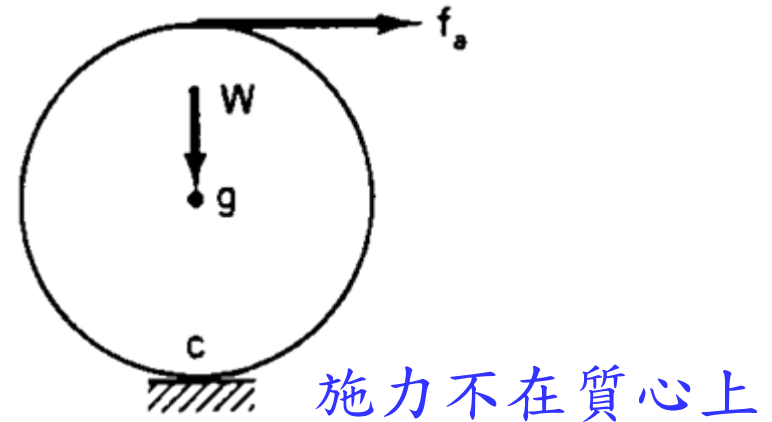
Rolling cylinder (cont.)

1) Using X as the generalized coordinate

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) - \frac{\partial L}{\partial X} = Q_x$$

$$L = T - V$$

=



The work input done by f_a is

$$dW =$$



Rolling cylinder (cont.)

The equation of motion is then:

2) Using θ as the generalized coordinate

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_{\theta} \quad \theta \text{ is in}$$

$$L = T - V =$$

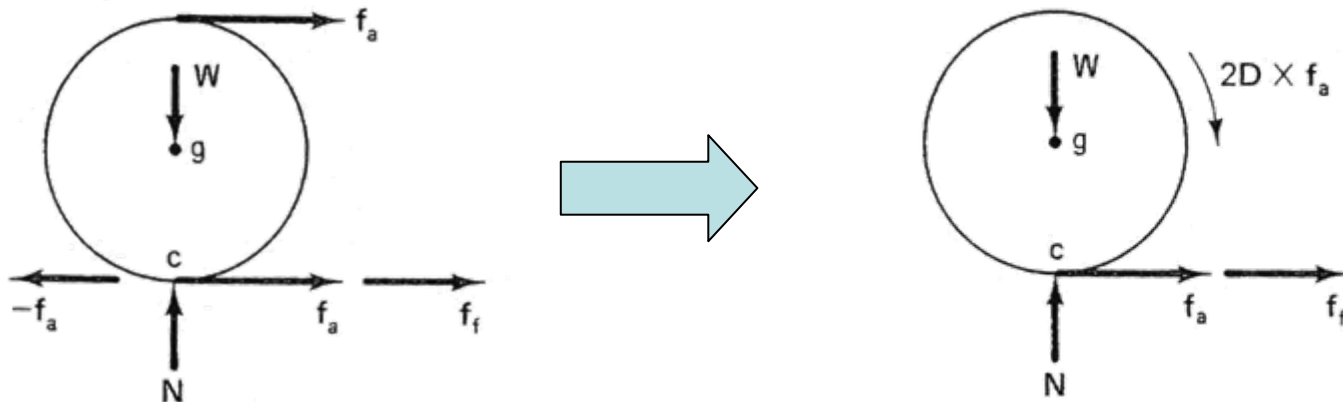
$$V = \text{ as before}$$



Rolling cylinder (cont.)

※ The work relative to θ is obtained by transferring to a

→ shift to the and a will be created by the effect.





Rolling cylinder (cont.)

\therefore and do not generate work

\rightarrow only the creates work on the system.

$$dW =$$

\therefore The equation of motion is then:



Rolling cylinder (cont.)

- Rolling with slip

∴ slip

→

does work

→

force still does no work

→

independent G.C. →

$T =$

$$L = T - V = T$$

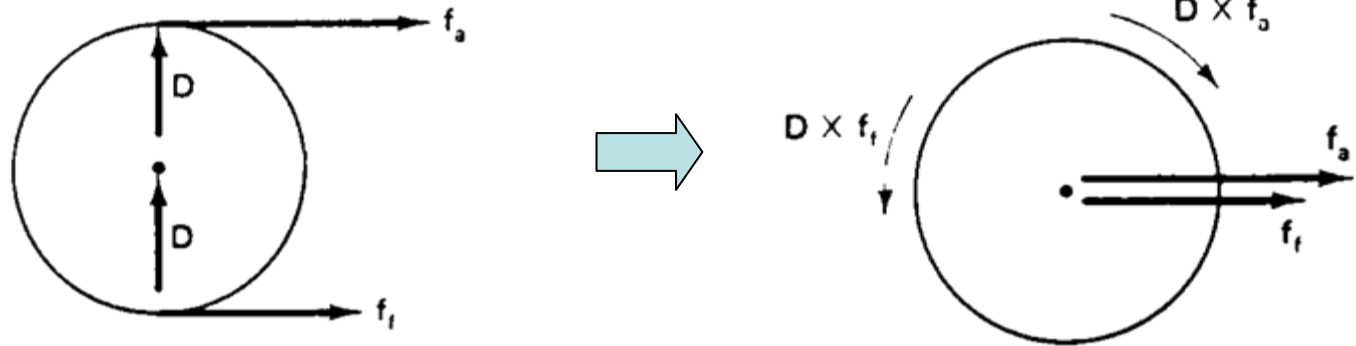
Two Lagrange equations of motion need to apply.



Rolling cylinder (cont.)

Derivation of generalized forces

→ Moving all forces and couples to



$$Q_x =$$

$$Q_\theta =$$



Rolling cylinder (cont.)

The equations of motion are:

in X direction \rightarrow

in θ direction \rightarrow

※ compare that derived in Chap.3:

$$-Df_a + \mu_d WD + I_{zz} \dot{\omega}_z = 0$$

Think:

\rightarrow Why they are different in sign?

\rightarrow How to determine whether it rolls or not by the Lagrange approach?



Pendulum

- Single pendulum (pin fixed) :
If the spring yields no torque when

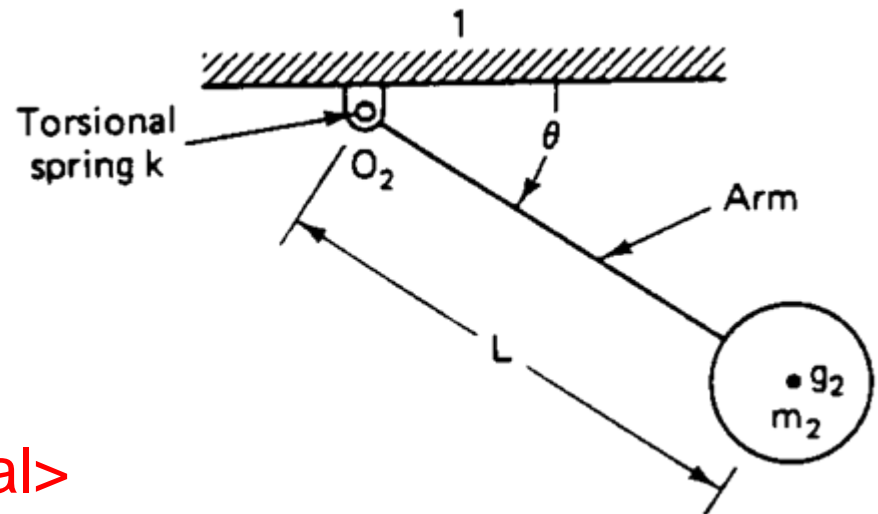
$V =$

<spring> <gravity>

$T =$

<translational> <rotational>

=





Pendulum (cont.)

Assume the arm is massless:

θ is the only generalised coordinate

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_{\theta} \quad L = T - V$$

The EOM is:

→ The moment of gravity is counterbalanced by the
and



Pendulum (cont.)

When $\omega = 0$, the static equilibrium location θ_0 is found by

→ If $\theta = \theta_0 + \varphi$, the EOM is

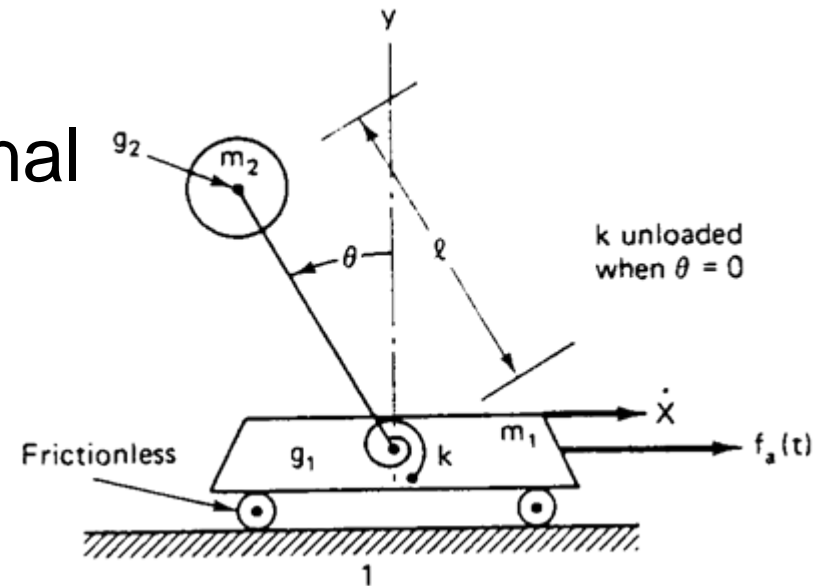
→ The incremental coordinates do not
the EOM when **terms**
exist.



Pendulum (cont.)

- Traveling inverted pendulum (pin translates)

When $\theta = 0$, the torsional spring k is unloaded



$$V_{g1} =$$

$$\therefore \vec{l} =$$

$$\dot{\vec{\theta}} = \dot{\theta} \vec{k}$$



Pendulum (cont.)

$$v_{g2} =$$

$$T = \frac{1}{2} m_1 v_{g1}^2 + \frac{1}{2} m_1 v_{g2}^2 + \frac{1}{2} I_{g2} \dot{\theta}^2$$
$$=$$

<translation
of >

<translation
of >

<rotation
of >

G.C. \Rightarrow

$$\theta_X = \quad , \quad Q_\theta =$$



Pendulum (cont.)

∴ The EOM are :

1) For G.C. of X :

⊗ combined comp. of inertia + horizontal and inertias.

2) For G.C. of θ :

⊗ combined moment of about + couple + (only on).



Pendulum (cont.)

- Rotating pendulum (pin rotates)

The mass “ m ” describes a **spherical motion**

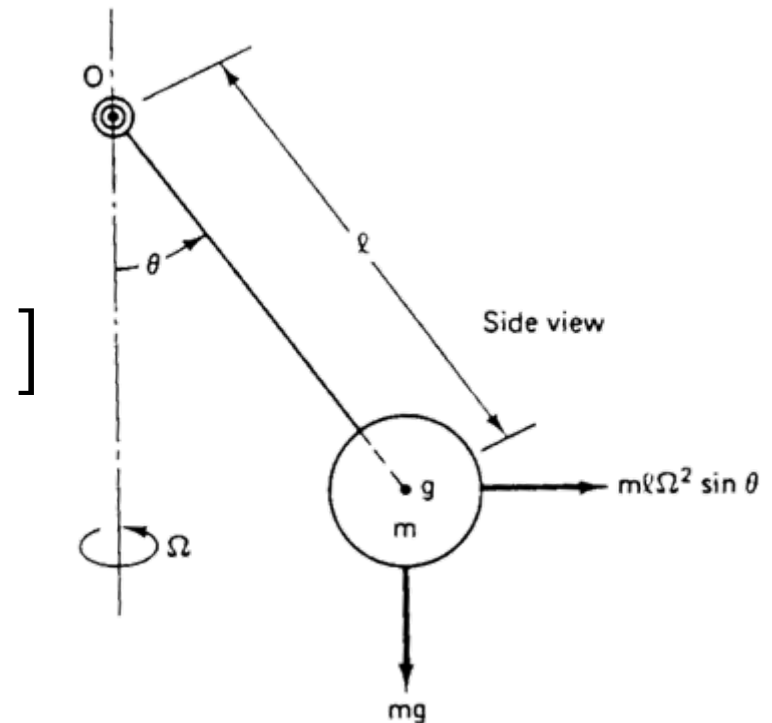
$$V_{long} =$$

$$V_{latit} =$$

$$T = \frac{1}{2} m \left[\right]$$

+

$$V =$$





Pendulum (cont.)

$G.C. \Rightarrow$

\therefore The EOM are :

1) For the G.C. of θ :

<move from
>
<moment of
force
about >
↓
<moment of
about
>

$$\ddot{\theta} = 0 \rightarrow$$

$$\therefore \Omega \uparrow,$$



Pendulum (cont.)

2) For G.C. of Ω :

※ For T_a from start-up and reduce to zero

$$\left[I_g + (l \sin \theta)^2 m \right] \Omega =$$

→ The constant $\left[I_g + (l \sin \theta)^2 m \right] \Omega$ can be derived.

if $\Omega = \text{const.} \Rightarrow$



The gyroscopic effect

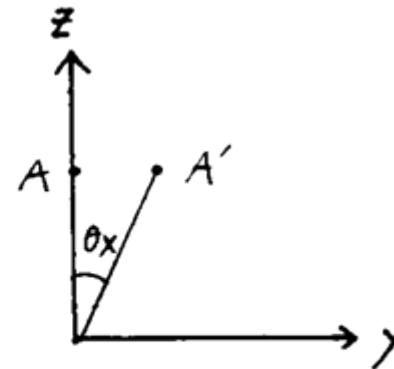
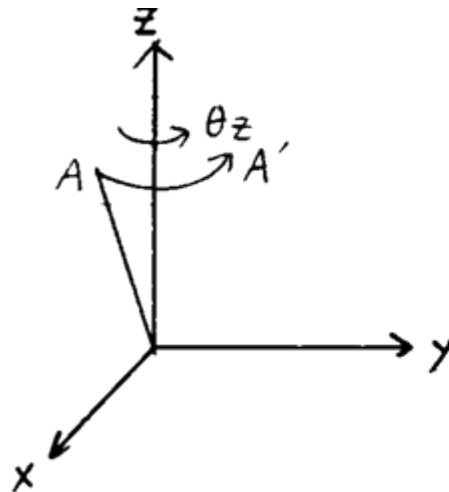
- Gyroscopic motion

1) The need of $\theta_x, \theta_y, \theta_z$ angles

→ For a **rotation** about **a**

fixed axis, the Cartesian coordinates X, Y, Z are no longer

e.g.





Gyroscopic effect (cont.)

2) Definitions of Euler angles

- ① rotate the body **about** **axis** through angle
- ② rotate the body about **axis** with an angle
- ③ rotate the body about **axis** with an angle

: angle of (公轉)

: angle of (下垂)

: angle of (自轉)

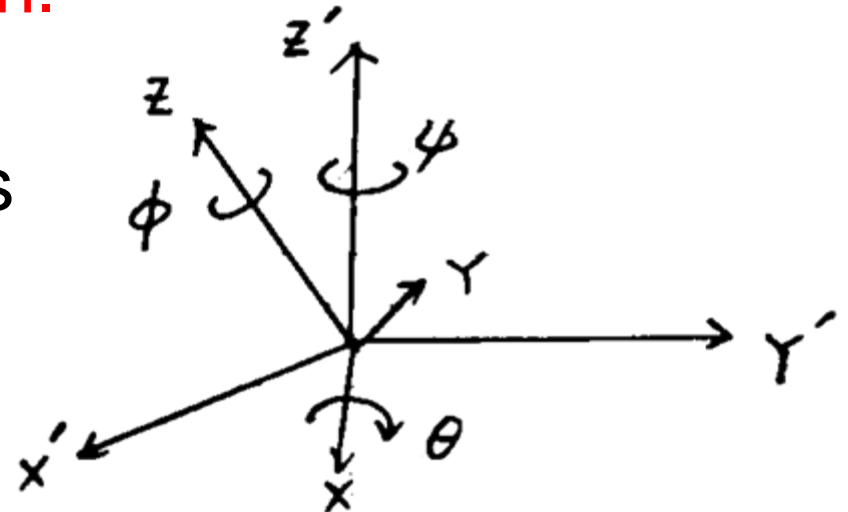


Gyroscopic effect (cont.)

3) Euler angles are (ϕ, θ, ψ) coordinates (required by (ϕ, θ, ψ)) but not (x, y, z)

4) Cartesian coordinates are (x, y, z) (required by (ϕ, θ, ψ)) but not (ϕ, θ, ψ) for (ϕ, θ, ψ) motion.

5) (ϕ, θ, ψ) are defined as one looks to the origin O along the rotation axis to see (ϕ, θ, ψ)





Example

- The spinning top

※ Choices of the coordinate system

1) The amount of

x, y, z must be axes.

→ the

2) The forms of

are along x & z () axes in

the xyz coordinate system.



Example (cont.)

By using xyz as the coordinate system, relationships between x, y, z and ψ, θ, ϕ are :

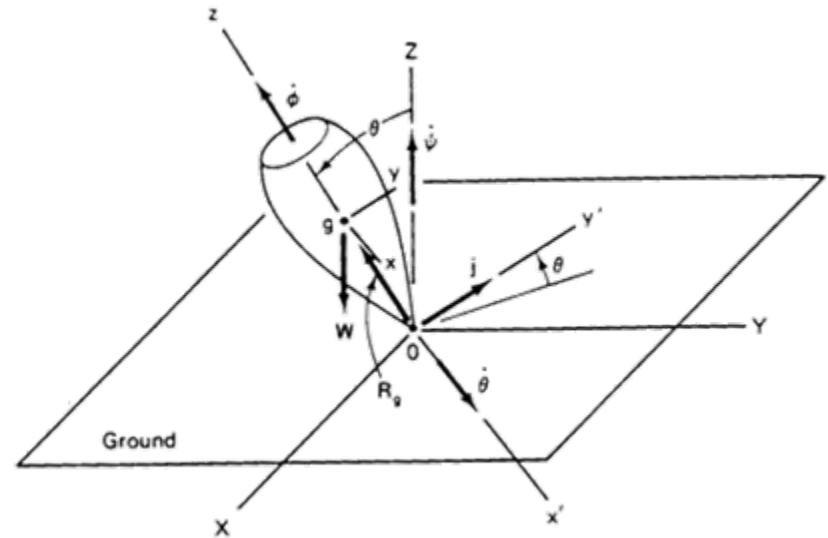
$$\omega_x =$$

$$\omega_y =$$

$$\omega_z =$$

The position of C.G. and its linear velocity can be derived below:

$$\vec{R}_g =$$





Example (cont.)

$$\vec{\Omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$$\dot{\vec{R}}_g =$$

∴ The total kinetic energy of this boy is:

$$T = \frac{1}{2} m \dot{\vec{R}}_g^2 + \frac{1}{2} [I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2]$$

$$= \frac{1}{2} m [\quad]$$
$$+ \frac{1}{2} [\quad]$$



Example (cont.)

$$= \frac{1}{2} \left[(I_{xx} + mR_g^2) \dot{\theta}^2 + (I_{xx} + mR_g^2) \cdot (\dot{\psi} \sin \theta)^2 + I_{zz} (\dot{\phi} + \dot{\psi}^2 \cos \theta)^2 \right]$$

By designating new $x'y'z$ coordinates with its origin at O & the same orientation as xyz

$$(\because I_{xx} = \quad , \quad I_{zz} = \quad , \quad I'_{xx} = \quad)$$

$$\therefore T = \frac{1}{2}$$

$$V = \quad , \quad L = T - V$$



Example (cont.)

G.C. :

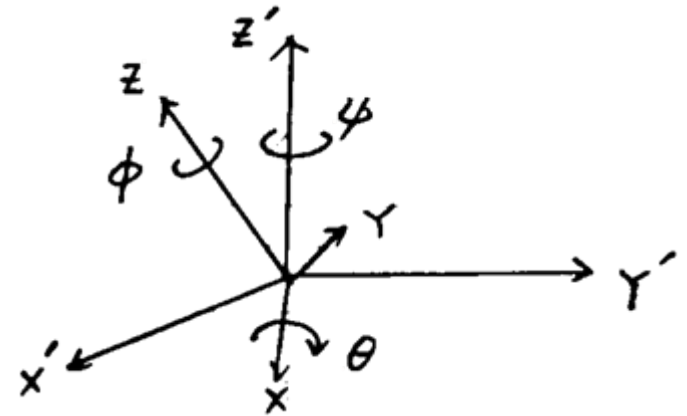
$$Q_\theta = \quad , \quad Q_\phi = \quad , \quad Q_\psi =$$

The EOM :

θ :

$$\phi : \frac{d}{dt} [\dot{\phi} + \dot{\psi} \cos \theta]$$

$$\psi : \frac{d}{dt} [I'_{xx} (\dot{\psi} \sin^2 \theta) + I'_{zz} (\dot{\phi} + \dot{\psi} \cos \theta) \cos \theta]$$





Example (cont.)

※ Discussions:

→ precession

→ a special case of “spinning top”

→ the conditions

1) A constant

2) A constant

3) A constant

The last two EOM become (both).



Example (cont.)

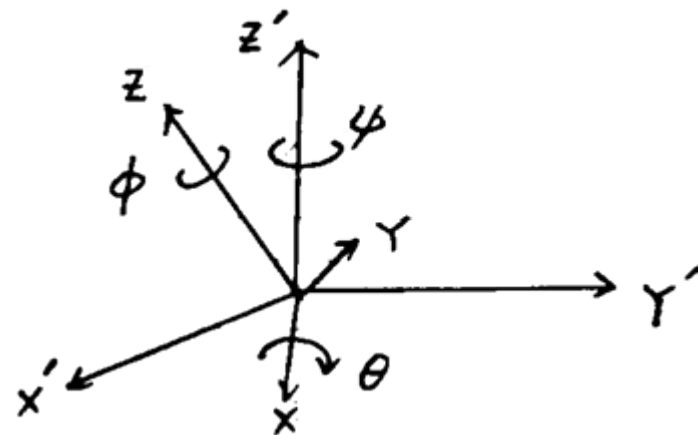
The first EOM is:

$$I'_{zz} \omega_z \dot{\psi} \sin \theta - I'_{xx} \dot{\psi}^2 \sin \theta \cos \theta =$$

$$\therefore \omega_z \downarrow ,$$

In particular, for $\theta = 90^\circ$

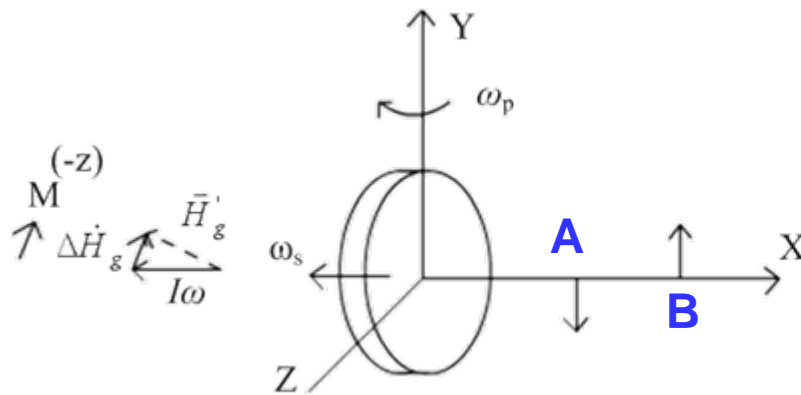
$$\therefore \omega_z \downarrow ,$$





Example (cont.)

- A simplified view of "steady precession"
 → a propeller shaft of an airplane



(方向× 方向)

⊗ Applied in direction, the system rotates in direction.



Example (cont.)

※ The gyroscopic effect is an extension of:

Change of ω = applied



與 同向

※ At this moment, bearings A & B must provide the disk with torque and **the airplane** because **the disk will exercise a** on the bearings.



Application

- The rate gyro

→ A gyro

For the rotor

$$I_{zz} = 3 \times 10^{-4} \text{ slug} - \text{ft}^2$$

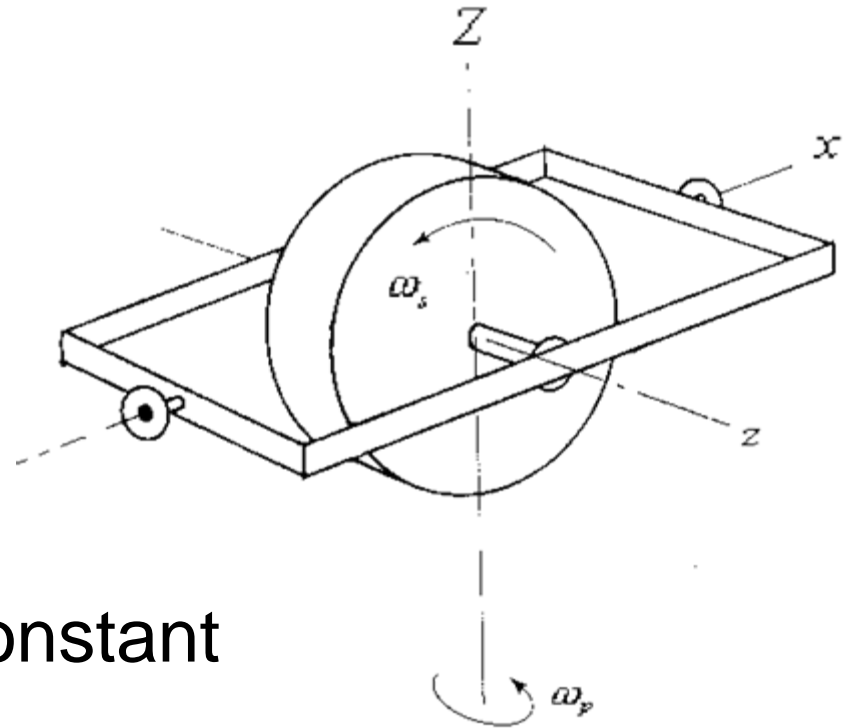
$$I_{xx} = 1.5 \times 10^{-4} \text{ slug} - \text{ft}^2$$

The combined spring constant

$$k_t = 4.95 \text{ ft} - \text{lb} / \text{rad}$$

The precession axis Z has

$$\omega_p = \dot{\psi} = 1 \text{ rad} / \text{sec}$$





Application (cont.)

The spin axis z has

$$\omega_s = \dot{\phi} = 2 \times 10^4 \quad \text{rad / sec}$$

For this case, $\dot{\psi} = 0$, but an external torque by the spring is applied.

Assume the spring is unstretched when $\theta = \frac{\pi}{2}$.

From

$$k_t \left(\frac{\pi}{2} - \theta \right) = I_{zz} (\dot{\phi} + \dot{\psi} \cos \theta) \dot{\psi} \sin \theta - I_{xx} \dot{\psi}^2 \sin \theta \cos \theta$$



Application (cont.)

→ Substitute given values into the equation

$$4.95\left(\frac{\pi}{2} - \theta\right) = 3 \times 10^{-4} (2 \times 10^4 + \cos \theta) \sin \theta - 1.5 \times 10^{-4} \sin \theta \cos \theta$$

By trial & error $\Rightarrow \theta =$

- in practice, usually use **a** to maintain an angle close to
- **measure** & by knowing , can be obtained.



Linearization

- Linearization of EOM

- 1) The method

- a) Assume θ is

e.g. $\sin \theta \approx \theta$, $\tan \theta \approx \theta$, $\cos \theta \approx 1 - \frac{\theta^2}{2}$

- b) Compare the θ^2 of terms

e.g. $ml^2 \theta^2$, $I_g \theta^2$, θ_0^2 , 1

- 2) The usages

- a) Reveal the θ^2 behaviors of the system.

- b) θ^2 calculation efforts.



Linearization (cont.)

- c) A to the computer simulation results when the conditions are met.
- d) Care should be taken on the situations of the parameter in question **is**
 - They can not or even the system behaviors.
- e) As **a** before performing detailed analysis.



END of Chap_4b