Machine Dynamics

Chapter 4a：Lagrange Equations

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Lagrange v.s. D'Alembert

- Comparison

Lagrange equations	D'Alembert's principle
approach (&)	approach (&)
method	method
Solve of dynamic problems	Solve only determined problem
Up to terms	Up to terms

Lagrange equation

- For a particle

  Newton's 2nd law,

  Transform into scalar equations

$$\sum_{j=1}^{n} (f_{ij})_x + (F_i)_x = \frac{d}{dt}(m_i \cdot \dot{X}_i) \qquad , etc.$$

1) K.E. of a particle in a Cartesian coordinates:

Lagrange equation (cont.)

2) Externally applied forces

Can be expressed by the	Can not
① force	① force
② force	② force

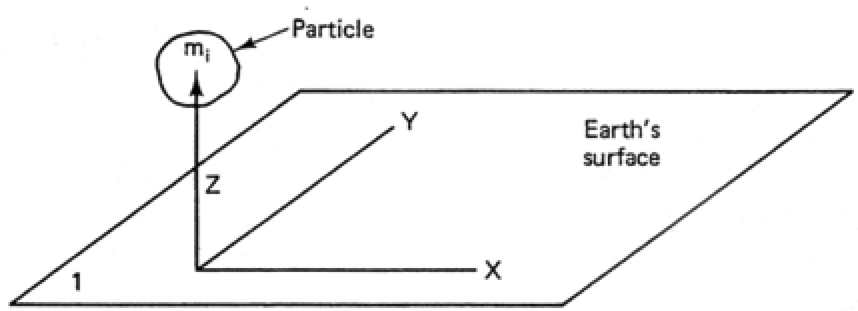
force

Lagrange equation (cont.)

※ gravitational force

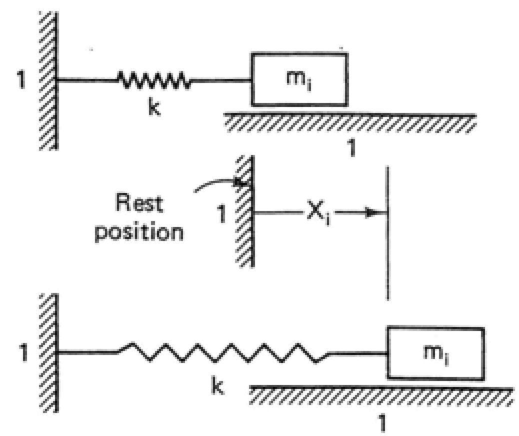
The potential energy of a particle at an elevation Z is:

$$V_{gi} = m_i g Z$$



※ elastic force

$$V_{si} = \frac{1}{2} k X_i^2$$



Lagrange equation (cont.)

In general

$$\vec{F}_{i,conservative} = -\nabla V_i =$$

$$\therefore \vec{F}_i = -\nabla V_i + \vec{Q}_i$$

→ These are the equations of motion for the

Lagrange equation (cont.)

- For a rigid body

From a particle to a rigid body

1. $\vec{f}_{ij} = -\vec{f}_{ji} \longrightarrow$

2. $\sum_{i=1}^{N} (Q_i)_X \longrightarrow$

3. $\sum_{i=1}^{N} V_i \longrightarrow$ (of the body's .)

4. $\sum_{i=1}^{N} T_i \longrightarrow$ (refer to .)

Lagrange equation (cont.)

So the Lagrange EOM for a rigid body is:

Assume

$$V = V(X, Y, Z) \quad , \quad T = T(\dot{X}, \dot{Y}, \dot{Z})$$

Define a

$$\frac{\partial L}{\partial \dot{X}} = \frac{\partial T}{\partial \dot{X}} \quad , \quad \frac{\partial L}{\partial X} = -\frac{\partial V}{\partial X} \quad , etc.$$

Lagrange equation (cont.)

The rigid body Lagrange equations are:

1) L : the stored in the system

which includes

$L = T - V$ is based on

$\rightarrow$ Why? See the appendix

2) : change in

Lagrange equation (cont.)

3) Q_x : forces cause the

to

→ forces which add or remove

from the system are

e.g. constraining forces like

force ($V =$)

force (V)

→ in the direction of forces

but force does

energy from the system.

Lagrange equation (cont.)

- Formulation of basic terms in Lagrange equations

  of a rigid body

  Treat the rigid body as a continuum, the total kinetic energy will be:

$$T =$$

Lagrange equation (cont.)

The velocity of *dm* is assigned as:

$$\dot{\vec{r}}_o = \dot{X}\vec{i} + \dot{Y}\vec{j} + \dot{Z}\vec{k}$$

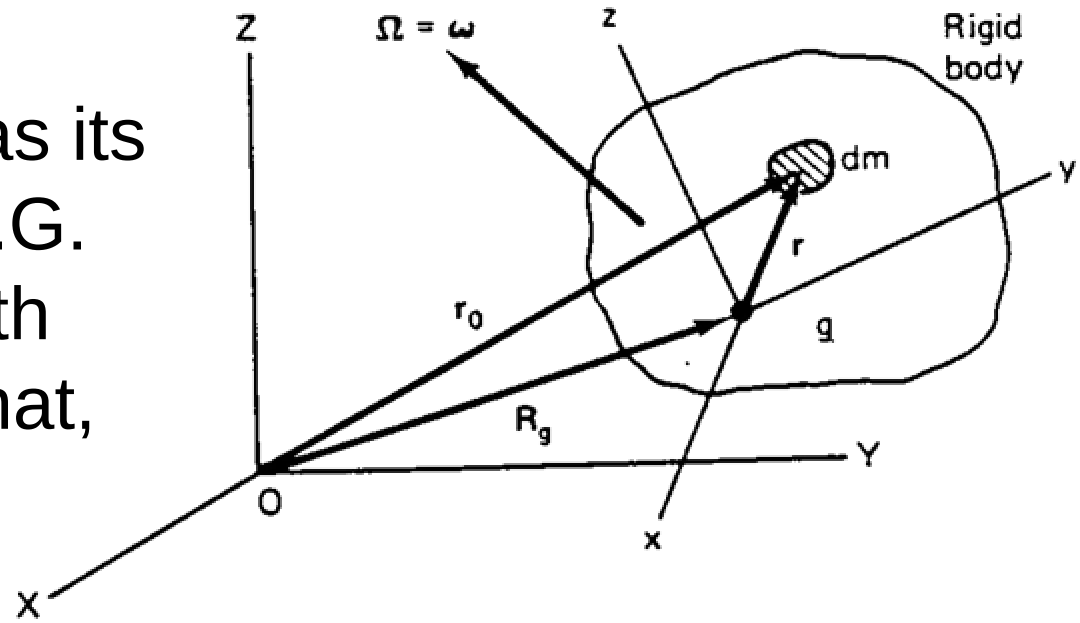
※Represent the in terms of the motion of its individual particles

→ transform to "in terms of the motion of its

Lagrange equation (cont.)

If the xyz coordinates has its origin at the C.G. and rotates with the body, so that,

$$\vec{\Omega} = \vec{\omega}$$



The position of *dm* is:

The velocity of *dm* is:

Lagrange equation (cont.)

$\vec{r}$ is a vector in the coordinate xyz (in rigid body, the particles relative velocity)

$$\therefore \dot{\vec{r}}_o \cdot \dot{\vec{r}}_o = \dot{\vec{R}}_g \cdot \dot{\vec{R}}_g + 2\dot{\vec{R}}_g \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})$$

$T =$ (1) (2)

$$+\frac{1}{2}\iiint [(\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r})(\vec{\omega} \times \vec{r})]\, dm$$

(3)

Lagrange equation (cont.)

(1)

※ The associated with the motion of

(2) $= \dot{\vec{R}}_g \cdot \left(\vec{\omega} \times \iiint \vec{r}\, dm\right)$

$$(\because \vec{r}_g = \frac{1}{m} \iiint \vec{r}\, dm)$$

※ By the convenient choice of the xyz coordinates to make

Lagrange equation (cont.)

(3)

※The kinetic energy refer to the

Therefore, the total kinetic energy of the rigid body is:

$$T =$$

Lagrange equation (cont.)

2) P.E. of a rigid body

a) gravitational force

$$V_g = m_g Z_g$$

b) elastic force

$$V_s = \frac{1}{2} k X_g^2$$

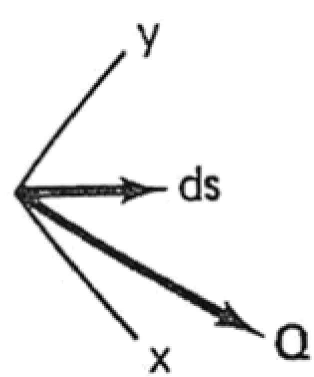
※ Both the K.E. and P.E. are referred to the

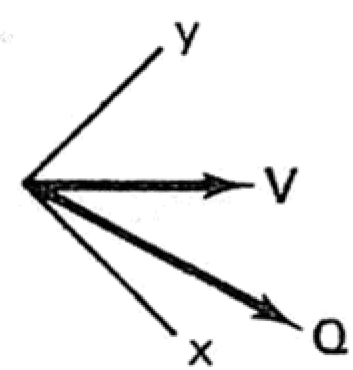
Lagrange equation (cont.)

3) Generalized force / couple

   a) a force does work to a system by a way

      & its rate:





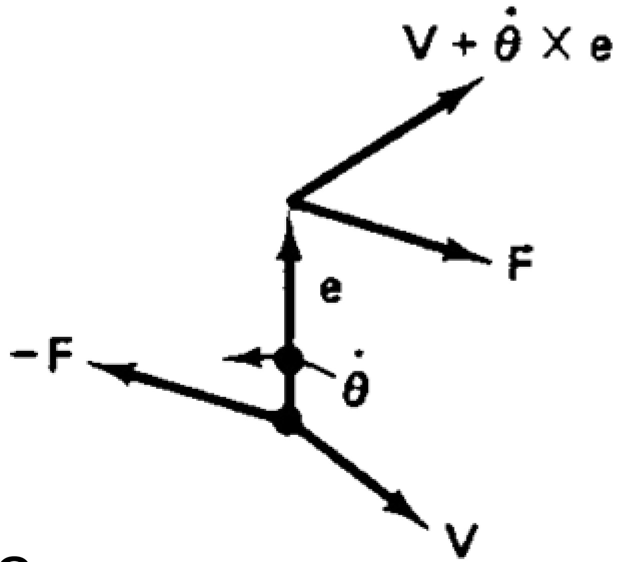
Lagrange equation (cont.)

b) for a couple

$$\frac{dw}{dt} =$$

$$= \vec{F} \cdot (\dot{\vec{\theta}} \times \vec{e})$$

$$=$$



※ is the couple corresponding to the

, etc.

Lagrange equation (cont.)

4) Generalized coordinates

a) The coordinates used to locate a dynamic system w.r.t. a

b) They must satisfy:

① :

→ they are to locate the configuration of in a system.

e.g. For a 3D point :

incomplete

complete

Lagrange equation (cont.)

② :

→ if one of them is fixed, there still remains a that the other one can take.

e.g. independent

dependent

c) There is Lagrange equation corresponding to each coordinates.

Lagrange equation (cont.)

d) The of each Q force or couple with an increment of its associated represents an of done by that Q upon the system.

e.g. Translational work

Rotational work

※The forms of are determined by the

END of Chap_4a