

$$1. \quad y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$\text{Let } y(0) = a_0, \quad y'(0) = a_1$$

$$(x^2 + 2x + 1)y'' = 2(x+1)y' - 2y$$

$$\text{apply } x=0 \Rightarrow y''(0) = 2y'(0) - 2y(0) = 2a_1 - 2a_0$$

Take differentiation:

$$(2x+2)y'' + (x^2+2x+1)y''' = 2y' + 2(x+1)y'' - 2y'$$

$$\Rightarrow (x^2+2x+1)y''' = 0 \Rightarrow y'''(0) = 0$$

Take differentiation again:

$$(2x+2)y''' + (x^2+2x+1)y^{(4)} = 0 \Rightarrow y^{(4)}(0) = -2y'''(0) = 0$$

$$\begin{aligned} \therefore y(x) &= a_0 + a_1x + (a_1 - a_0)x^2 + 0 \\ &= \underline{a_0(1-x^2) + a_1(x+x^2)} \end{aligned}$$

$$2. \quad \text{Change of Variables: } t = x-2$$

$$\text{Chain Rule: } \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt}(1) = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dt} \right) = \frac{d}{dt} \left(\frac{dy}{dt} \right) \frac{dt}{dx} = \frac{d^2y}{dt^2}(1) = \frac{d^2y}{dt^2}$$

$$y'' - (x-2)y' + 2y = 0 \Rightarrow \frac{d^2y}{dt^2} - t \frac{dy}{dt} + 2y = 0$$

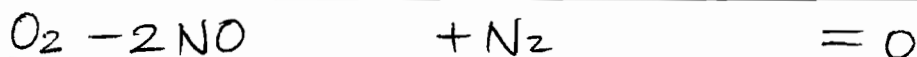
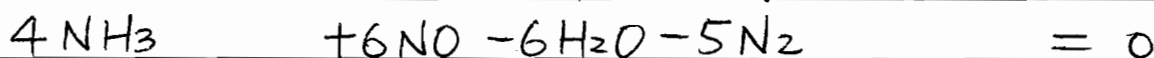
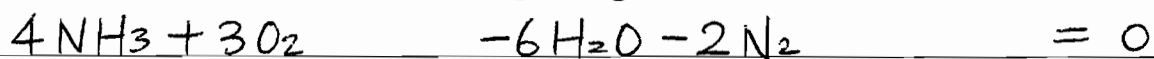
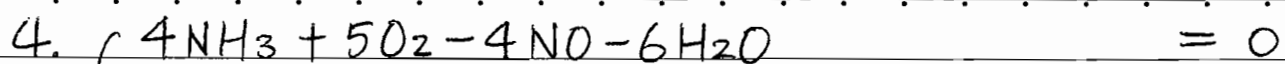
and this equation is to be solved near $t=0$.

From the class example:

$$y = a_0(1-t^2) + a_1 \left(t - \frac{1}{6}t^3 - \frac{1}{120}t^5 - \frac{1}{1680}t^7 - \dots \right)$$

Substituting $t = x-2$ into this equation.

$$y = a_0[1-(x-2)^2] + a_1 \left[(x-2) - \frac{1}{6}(x-2)^3 - \frac{1}{120}(x-2)^5 - \frac{1}{1680}(x-2)^7 - \dots \right]$$



$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 4 & 3 & 0 & -6 & -2 & 0 \\ 4 & 0 & 6 & -6 & -5 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \text{NH}_3 \\ \text{O}_2 \\ \text{NO} \\ \text{H}_2\text{O} \\ \text{N}_2 \\ \text{NO}_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -5 & 10 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \text{NH}_3 \\ \text{O}_2 \\ \text{NO} \\ \text{H}_2\text{O} \\ \text{N}_2 \\ \text{NO}_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \text{NH}_3 \\ \text{O}_2 \\ \text{NO} \\ \text{H}_2\text{O} \\ \text{N}_2 \\ \text{NO}_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \text{NH}_3 \\ \text{O}_2 \\ \text{NO} \\ \text{H}_2\text{O} \\ \text{N}_2 \\ \text{NO}_2 \end{bmatrix} = 0$$

∴ 3 independent chemical reactions

$$5. |A - \lambda I| = -\lambda^3 + 8\lambda^2 - 13\lambda + 6 = 0$$

$$\text{Caley-Hamilton: } -A^3 + 8A^2 - 13A + 6I = 0$$

(1) Multiply A^{-1} on both sides:

$$-A^2 + 8A - 13I + 6A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{6}(A^2 - 8A + 13I)$$

$$= \frac{1}{6} \begin{bmatrix} 11 & -5 & -5 \\ 5 & 1 & -5 \\ 5 & -5 & 1 \end{bmatrix} *$$

Note =

$$A^2 = \begin{bmatrix} -34 & 35 & 35 \\ -35 & 36 & 35 \\ -35 & 35 & 36 \end{bmatrix}$$

$$(2) A^3 = 8A^2 - 13A + 6I$$

$$A^4 = A(8A^2 - 13A + 6I)$$

$$= 8A^3 - 13A^2 + 6A$$

$$= 8(8A^2 - 13A + 6I) - 13A^2 + 6A$$

$$= 51A^2 - 98A + 48I$$

$$A^5 = A(51A^2 - 98A + 48I)$$

$$= 51A^3 - 98A^2 + 48A$$

$$= 51(8A^2 - 13A + 6I) - 98A^2 + 48A$$

$$= 310A^2 - 615A + 306I$$

$$= 310 \begin{bmatrix} -34 & 35 & 35 \\ -35 & 36 & 35 \\ -35 & 35 & 36 \end{bmatrix} - 615 \begin{bmatrix} -4 & 5 & 5 \\ -5 & 6 & 5 \\ -5 & 5 & 6 \end{bmatrix} + 306 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10546 & 10850 & 10850 \\ -10850 & 11160 & 10850 \\ -10850 & 10850 & 11160 \end{bmatrix} - \begin{bmatrix} -2460 & 3075 & 3075 \\ -3075 & 3690 & 3075 \\ -3075 & 3075 & 3690 \end{bmatrix} + \begin{bmatrix} 306 & 0 & 0 \\ 0 & 306 & 0 \\ 0 & 0 & 306 \end{bmatrix}$$

$$= \begin{bmatrix} -7774 & 7775 & 7775 \\ -7775 & 7776 & 7775 \\ -7775 & 7775 & 7776 \end{bmatrix} *$$

$$6. (A - \lambda I)X = 0 \Rightarrow \lambda = -2, 9, -18 \text{ (eigenvectors)}$$

$$(1) \lambda = -2 \quad \begin{cases} 4x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 + 4x_2 - 6x_3 = 0 \\ -6x_1 - 6x_2 - 13x_3 = 0 \end{cases} \Rightarrow x_1 = -x_2, x_3 = 0$$

$$\therefore X_1 = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$(2) \lambda = 9 \quad \begin{cases} -7x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 - 7x_2 - 6x_3 = 0 \\ -6x_1 - 6x_2 - 24x_3 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 = 2 : 2 : -1$$

$$\therefore X_2 = c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$(3) \lambda = -18 \quad \begin{cases} 20x_1 + 4x_2 - 6x_3 = 0 \\ 4x_1 + 20x_2 - 6x_3 = 0 \\ -6x_1 - 6x_2 + 3x_3 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 = 1 : 1 : 4$$

$$\therefore X_3 = c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\text{Eigenvectors} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\text{Orthogonality} = X_j^T \cdot X_i = 0$$

$$X_1^T \cdot X_2 = 2 - 2 = 0, \therefore X_1 \text{ \& } X_2 \text{ orthogonal}$$

$$X_2^T \cdot X_3 = 2 + 2 - 4 = 0, \therefore X_2 \text{ \& } X_3 \text{ orthogonal}$$

$$X_3^T \cdot X_1 = 1 - 1 = 0, \therefore X_3 \text{ \& } X_1 \text{ orthogonal}$$