Engineering Math I
Final

1. $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots\cdots$

Let $y(0) = a_0$, $y'(0) = a_1$

$(x^2+2x+1)y'' = 2(x+1)y' - 2y$

apply $x=0 \Rightarrow y''(0) = 2y'(0) - 2y(0) = 2a_1 - 2a_0$

Take differentiation:

$(2x+\cancel{2})y'' + (x^2+2x+1)y''' = \cancel{2y'} + 2(\cancel{x+1})y'' - \cancel{2y'}$

$\Rightarrow (x^2+2x+1)y''' = 0 \Rightarrow y'''(0) = 0$

Take differentiation again:

$(2x+2)y''' + (x^2+2x+1)y^{(4)} = 0 \Rightarrow y^{(4)}(0) = -2y'''(0) = 0$

$\therefore y(x) = a_0 + a_1 x + (a_1 - a_0)x^2 + 0$

$= \underline{a_0(1-x^2) + a_1(x+x^2)}$ ※

2. Change of Variables: $t = x-2$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}(1) = \frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d^2y}{dt^2}(1) = \frac{d^2y}{dt^2}$

$y'' - (x-2)y' + 2y = 0 \Rightarrow \frac{d^2y}{dt^2} - t\frac{dy}{dt} + 2y = 0$

and this equation is to be solved near $t=0$.

From the class example:

$y = a_0(1-t^2) + a_1\left(t - \frac{1}{6}t^3 - \frac{1}{120}t^5 - \frac{1}{1680}t^7 - \cdots\cdots\right)$

Substituting $t = x-2$ into this equation.

$\underline{y = a_0[1-(x-2)^2] + a_1\left[(x-2) - \frac{1}{6}(x-2)^3 - \frac{1}{120}(x-2)^5 - \frac{1}{1680}(x-2)^7 - \cdots\right]}$ ※

4.
$$\begin{cases} 4NH_3 + 5O_2 - 4NO - 6H_2O = 0 \\ 4NH_3 + 3O_2 - 6H_2O - 2N_2 = 0 \\ 4NH_3 + 6NO - 6H_2O - 5N_2 = 0 \\ O_2 + 2NO - 2NO_2 = 0 \\ -O_2 + 2NO - N_2 = 0 \\ O_2 - 2NO + N_2 = 0 \end{cases}$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 4 & 3 & 0 & -6 & -2 & 0 \\ 4 & 0 & 6 & -6 & -5 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} NH_3 \\ O_2 \\ NO \\ H_2O \\ N_2 \\ NO_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ 0 & -5 & 10 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} NH_3 \\ O_2 \\ NO \\ H_2O \\ N_2 \\ NO_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & -2 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} NH_3 \\ O_2 \\ NO \\ H_2O \\ N_2 \\ NO_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & -4 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} NH_3 \\ O_2 \\ NO \\ H_2O \\ N_2 \\ NO_2 \end{bmatrix} = 0$$

$\therefore$ 3 independent chemical reactions

5. $|A-\lambda I| = -\lambda^3+8\lambda^2-13\lambda+6=0$

Carley-Hamilton: $-A^3+8A^2-13A+6I=0$

(1) Multiply A^{-1} on both sides:

$$-A^2+8A-13I+6A^{-1}=0$$

$$\therefore A^{-1}=\frac{1}{6}(A^2-8A+13I)$$

$$=\frac{1}{6}\begin{bmatrix}11 & -5 & -5\\ 5 & 1 & -5\\ 5 & -5 & 1\end{bmatrix} \text{※}$$

Note:

$$A^2=\begin{bmatrix}-34 & 35 & 35\\ -35 & 36 & 35\\ -35 & 35 & 36\end{bmatrix}$$

(2)

$$A^3 = 8A^2-13A+6I$$

$$A^4 = A(8A^2-13A+6I)$$

$$= 8A^3-13A^2+6A$$

$$= 8(8A^2-13A+6I)-13A^2+6A$$

$$= 51A^2-98A+48I$$

$$A^5 = A(51A^2-98A+48I)$$

$$= 51A^3-98A^2+48A$$

$$= 51(8A^2-13A+6I)-98A^2+48A$$

$$= 310A^2-615A+306I$$

$$= 310\begin{bmatrix}-34 & 35 & 35\\ -35 & 36 & 35\\ -35 & 35 & 36\end{bmatrix} - 615\begin{bmatrix}-4 & 5 & 5\\ -5 & 6 & 5\\ -5 & 5 & 6\end{bmatrix} + 306\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-10540 & 10850 & 10850\\ -10850 & 11160 & 10850\\ -10850 & 10850 & 11160\end{bmatrix} - \begin{bmatrix}-2460 & 3075 & 3075\\ -3075 & 3690 & 3075\\ -3075 & 3075 & 3690\end{bmatrix} + \begin{bmatrix}306 & 0 & 0\\ 0 & 306 & 0\\ 0 & 0 & 306\end{bmatrix}$$

$$= \begin{bmatrix}-7774 & 7775 & 7775\\ -7775 & 7776 & 7775\\ -7775 & 7775 & 7776\end{bmatrix} \text{※}$$

6. $(A-\lambda I)X=0 \Rightarrow \lambda=-2,9,-18$ ※ (eigenvectors)

(1) $\lambda=-2$
$$\begin{cases} 4x_1+4x_2-6x_3=0 \\ 4x_1+4x_2-6x_3=0 \\ -6x_1-6x_2-13x_3=0 \end{cases} \Rightarrow x_1=-x_2,\ x_3=0$$

$$\therefore X_1 = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

(2) $\lambda=9$
$$\begin{cases} -7x_1+4x_2-6x_3=0 \\ 4x_1-7x_2-6x_3=0 \\ -6x_1-6x_2-24x_3=0 \end{cases} \Rightarrow x_1:x_2:x_3=2:2:-1$$

$$\therefore X_2 = c_2\begin{bmatrix}2\\2\\-1\end{bmatrix}$$

(3) $\lambda=-18$
$$\begin{cases} 20x_1+4x_2-6x_3=0 \\ 4x_1+20x_2-6x_3=0 \\ -6x_1-6x_2+3x_3=0 \end{cases} \Rightarrow x_1:x_2:x_3=1:1:4$$

$$\therefore X_3 = c_3\begin{bmatrix}1\\1\\4\end{bmatrix}$$

Eigenvectors: $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}$ ※

Orthogonality: $X_j^T \cdot X_i = 0$

$X_1^T \cdot X_2 = 2-2=0$, $\therefore$ X_1 & X_2 orthogonal
$X_2^T \cdot X_3 = 2+2-4=0$, $\therefore$ X_2 & X_3 orthogonal
$X_3^T \cdot X_1 = 1-1=0$, $\therefore$ X_3 & X_1 orthogonal