Appendix A: Modelling of the TWIP system

We derive the model of the TWIP system by two methods: the Newton Second Law and the Lagrange method.

A.1. Motor dynamics

The output torques of the motors can be represented as follows:

$$T_r = n_1\left[\frac{K_t}{Ls+R}\left(V_r - K_e\dot{\theta}_m^r\right)\right] - f_{bw}\dot{\theta}_m^r \tag{A1}$$

$$T_l = n_1\left[\frac{K_t}{Ls+R}\left(V_l - K_e\dot{\theta}_m^l\right)\right] - f_{bw}\dot{\theta}_m^l \tag{A2}$$

in which T_r and T_l are the output torques on the right and left, respectively. V_r and V_l are the input voltages on the right and left, respectively. n_1 is the gear ratio, K_t is the motor constant, while K_e is the DC motor back EMF constant. L and R are the motor inductance and resistance, respectively. θ_m^r and θ_m^l are the rotational angles of the right motor and the left motor, respectively. f_{bw} is friction coefficient between the cart body and the motors. Suppose the motor inertia is J_m, the applied torques to the wheels can be represented as follows:

$$T_L^r = T_r - n_1^2 J_m \ddot{\theta}_m^r \tag{A3}$$

$$T_L^l = T_l - n_1^2 J_m \ddot{\theta}_m^l \tag{A4}$$

where T_L^r and T_L^l are the torques on the right wheel and left wheel, respectively.

A.2. Dynamic Equations by the Newton's Second Law

First, we apply the Newton's Second Law with the assumption that the wheels roll without slipping. The schematic diagrams of the TWIP system is shown in Figure 1, where the rotational angle of the motor θ_m and the pitch angle ψ can be obtained from the encoder and the IMU, respectively. The wheels' angle can be derived as follows:

$$\begin{cases}\theta_r = \psi + \theta_m^r \\ \theta_l = \psi + \theta_m^l\end{cases} \tag{A5}$$

where θ_r and θ_l represent the rotational angles of the right and left wheels respectively and θ_m^r and θ_m^l represent the rotational angles of the right motor and the left motor, respectively. The forward wheel angle θ, the motor angle θ_m, and the steering angle ϕ are defined as:

$$\theta = \frac{1}{2}\left(\theta_r + \theta_l\right) \tag{A6}$$

$$\theta_m = \frac{1}{2}\left(\theta_m^r + \theta_m^l\right) \tag{A7}$$

$$\phi = \frac{r}{W}\left(\theta_r - \theta_l\right) \tag{A8}$$

Referring to Figure A1, the positions of points A_m, A_r, A_l, and B can be represented as follows:

$$
\begin{aligned}
&\vec{A}_m = (x_m\ , y_m\ , 0)\\
&\vec{A}_r = (x_r\ , y_r\ , 0) = (x_m + \frac{W}{2}\sin\phi, y_m - \frac{W}{2}\cos\phi\ , 0)\\
&\vec{A}_l = (x_l\ , y_l\ , 0) = (x_m - \frac{W}{2}\sin\phi, y_m + \frac{W}{2}\cos\phi\ , 0)\\
&\vec{B} = (x_p\ , y_p\ , z_p) = (x_m\ + \ell\sin\psi\cos\phi, y_m + \ell\sin\psi\sin\phi, \ell\cos\psi)
\end{aligned}
\tag{A9}
$$

Taking derivatives of (A9) gives the following equations:

$$
\begin{aligned}
&\dot{\vec{A}}_m = (r\dot{\theta}\cos\phi\ , r\dot{\theta}\sin\phi\ , 0)\\
&\dot{\vec{A}}_r = (\dot{x}_m + \frac{W}{2}\dot{\phi}\cos\phi, \dot{y}_m + \frac{W}{2}\dot{\phi}\sin\phi\ , 0)\\
&\dot{\vec{A}}_l = (\dot{x}_m - \frac{W}{2}\dot{\phi}\cos\phi, \dot{y}_m - \frac{W}{2}\dot{\phi}\sin\phi\ , 0)\\
&\dot{\vec{B}} = (\dot{x}_m + \ell\dot{\psi}\cos\psi\cos\phi - \ell\sin\psi\dot{\phi}\sin\phi, \dot{y}_m + \ell\dot{\psi}\cos\psi\sin\phi + \ell\sin\psi\dot{\phi}\cos\phi, -\ell\dot{\psi}\sin\psi)
\end{aligned}
\tag{A10}
$$

Similarly, taking derivatives of (A10) results in:

$$
\begin{aligned}
&\ddot{\vec{A}}_m = (r\ddot{\theta}\cos\phi - r\dot{\theta}\dot{\phi}\sin\phi, r\ddot{\theta}\sin\phi + r\dot{\theta}\dot{\phi}\cos\phi, 0)\\
&\ddot{\vec{A}}_r = (\ddot{x}_m + \frac{W}{2}\ddot{\phi}\cos\phi - \frac{W}{2}\dot{\phi}^2\sin\phi, \ddot{y}_m + \frac{W}{2}\ddot{\phi}\sin\phi + \frac{W}{2}\dot{\phi}^2\cos\phi\ , 0)\\
&\ddot{\vec{A}}_l = (\ddot{x}_m - \frac{W}{2}\ddot{\phi}\cos\phi + \frac{W}{2}\dot{\phi}^2\sin\phi, \ddot{y}_m - \frac{W}{2}\ddot{\phi}\sin\phi - \frac{W}{2}\dot{\phi}^2\cos\phi\ , 0)\\
&\ddot{\vec{B}} = (\ddot{x}_m + \ell\ddot{\psi}\cos\psi\cos\phi - \ell\dot{\psi}^2\sin\psi\cos\phi - 2\ell\dot{\psi}\cos\psi\dot{\phi}\sin\phi - \ell\sin\psi\ddot{\phi}\sin\phi - \ell\sin\psi\dot{\phi}^2\cos\phi,\\
&\qquad \ddot{y}_m + \ell\ddot{\psi}\cos\psi\sin\phi - \ell\dot{\psi}^2\sin\psi\sin\phi + 2\ell\dot{\psi}\cos\psi\dot{\phi}\cos\phi + \ell\sin\psi\ddot{\phi}\cos\phi - \ell\sin\psi\dot{\phi}^2\sin\phi,\\
&\qquad -\ell\ddot{\psi}\sin\psi - \ell\dot{\psi}^2\cos\psi)
\end{aligned}
\tag{A11}
$$

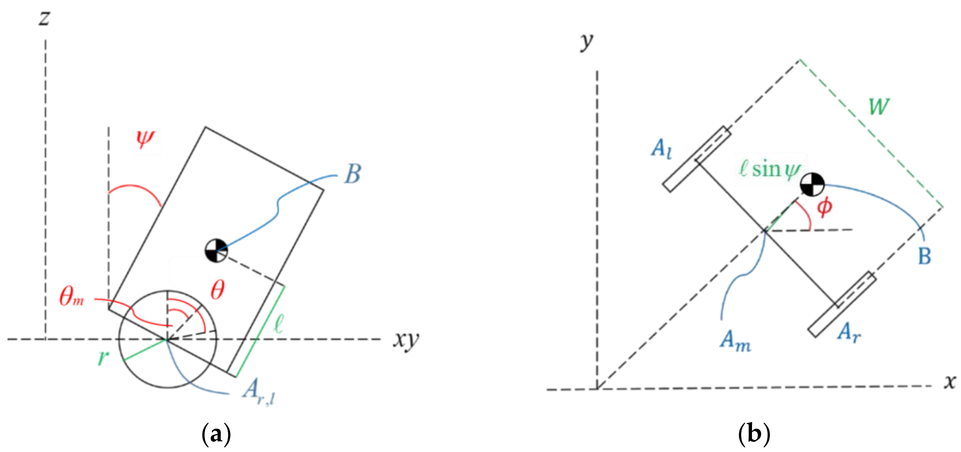

Figure A1. The TWIP 3-D system: (**a**) The schematic diagram in xy-z plane; (**b**) The schematic diagram in x-y plane.

The free body diagram of the left and right wheel are shown in Figure A2a,b with the corresponding D'Alembert's forces ($m\ddot{\vec{A}}_{l(x)}$, $m\ddot{\vec{A}}_{l(z)}$, $m\ddot{\vec{A}}_{r(x)}$, and $m\ddot{\vec{A}}_{r(z)}$) and the D'Alembert's moments ($J_w\ddot{\theta}_l$ and $J_w\ddot{\theta}_r$). The dynamic equations of the wheels can be expressed as follows:

$$
\sum F_x = 0 \text{ gives: } \begin{cases} f_s - F_{A_r(x)} - m\ddot{\vec{A}}_{r(x)} = 0 \\ f_s - F_{A_l(x)} - m\ddot{\vec{A}}_{l(x)} = 0 \end{cases}
\tag{A12}
$$

$$\sum F_z = 0 \text{ gives: } \begin{cases} N - mg - F_{A_r(z)} - m\ddot{A}_{r(z)} = 0 \\ N - mg - F_{A_l(z)} - m\ddot{A}_{l(z)} = 0 \end{cases} \tag{A13}$$

$$\sum M_C = 0 \text{ gives: } \begin{cases} rF_{A_r(x)} + rm\ddot{A}_{r(x)} + J_w\ddot{\theta}_r - T_L^r = 0 \\ rF_{A_l(x)} + rm\ddot{A}_{l(x)} + J_w\ddot{\theta}_l - T_L^l = 0 \end{cases} \tag{A14}$$

Similarly, the free body diagram of the cart is shown in Figure A2c with the corresponding D'Alembert's forces ($M\ddot{B}_x$ and $M\ddot{B}_z$), and the D'Alembert's moment $J_\psi\ddot{\psi}$. The dynamic equations can be represented as follows:

$$\sum F_x = 0 \text{ gives: } F_{A_r(x)} + F_{A_l(x)} - M\ddot{B}_x = 0 \tag{A15}$$

$$\sum F_z = 0 \text{ gives: } F_{A_r(z)} + F_{A_l(z)} - Mg - M\ddot{B}_z = 0 \tag{A16}$$

$$\sum M_{A_m} = 0 \text{ gives: } T_L^r + T_L^l - J_\psi\ddot{\psi} - Mg\ell\sin\psi - M\ddot{B}_z\ell\sin\psi + M\ddot{B}_x\ell\cos\psi = 0 \tag{A17}$$

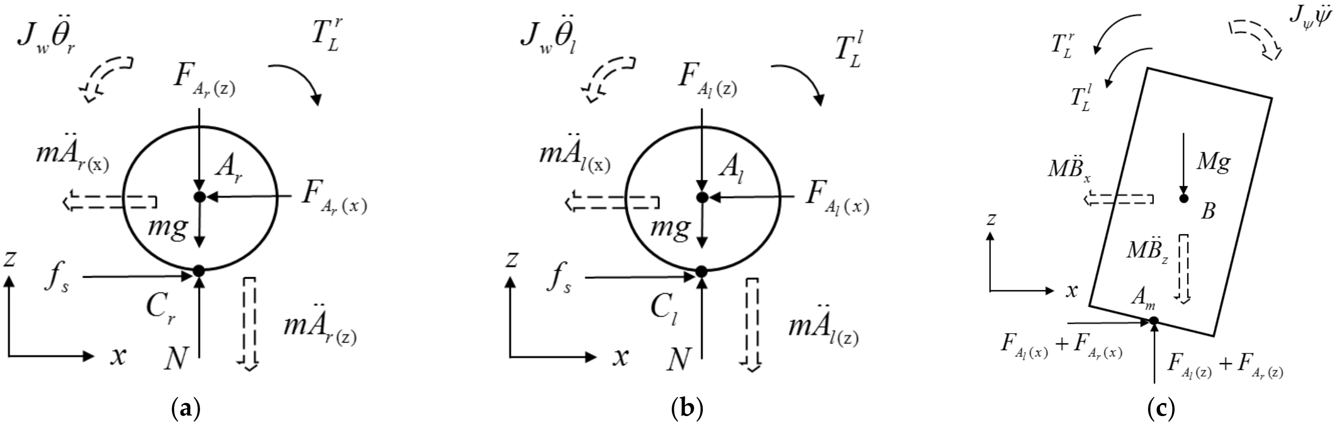

Figure A2. The free body diagram of TWIP in xy-z plane: (**a**) the right wheel; (**b**) the left wheel; (**c**) the cart.

Summarizing the two equations of (A14) and simplifying it with the (A3-A6)(A11)(A15) gives:

$$T_r + T_l = \left[(2m+M)r^2 + 2J_w + 2n_1^2 J_m\right]\ddot{\theta} + \left(Mr\ell\cos\psi - 2n_1^2 J_m\right)\ddot{\psi} - Mr\ell\dot{\psi}^2\sin\psi \tag{A18}$$

Taking linearization of (A18) about $\psi = 0$ results in:

$$T_r + T_l = \left[(2m+M)r^2 + 2J_w + 2n_1^2 J_m\right]\ddot{\theta} + \left(Mr\ell - 2n_1^2 J_m\right)\ddot{\psi} \tag{A19}$$

Substituting (A3-A6)(A11) to (A17) gives:

$$-(T_r + T_l) = \left(Mr\ell\cos\psi - 2n_1^2 J_m\right)\ddot{\theta} + \left(M\ell^2 + J_\psi + 2n_1^2 J_m\right)\ddot{\psi} - Mg\ell\sin\psi - M\ell^2\sin\psi\cos\psi\dot{\phi}^2 \tag{A20}$$

Taking linearization of (A20) about $\psi = 0$ results in:

$$-(T_r + T_l) = \left(Mr\ell - 2n_1^2 J_m\right)\ddot{\theta} + \left(M\ell^2 + J_\psi + 2n_1^2 J_m\right)\ddot{\psi} - Mg\ell\psi \tag{A21}$$

The free body diagram of the cart in the x-y plane is shown in Figure A3 with the corresponding D'Alembert's forces ($M\ddot{B}_x$ and $M\ddot{B}_y$) and the D'Alembert's moment $J_\phi\ddot{\phi}$. Taking moments about A_m gives:

$$\frac{W}{2}\left(F_{A_r(x)} - F_{A_l(x)}\right) - J_\phi\ddot{\phi} - M\ddot{B}_y\ell\sin\psi = 0 \tag{A22}$$

Substituting (A3-A6)(A11)(A14) to (A22) gives:

$$\frac{W}{2r}(T_r - T_l) = \left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2J_m\right) + M\ell^2\sin^2\psi\right]\ddot{\phi} + 2M\ell^2\dot{\psi}\sin\psi\cos\psi\dot{\phi} \tag{A23}$$

Taking linearization about $\phi = 0$ results in:

$$\frac{W}{2r}(T_r - T_l) = \left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2J_m\right)\right]\ddot{\phi} \tag{A24}$$

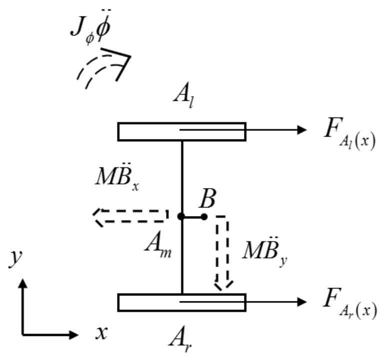

Figure A3. The free body diagram of TWIP in x-y plane.

A.3. Dynamic Equations by the Lagrange method

Now we apply the Lagrange method to derive the transfer function of the TWIP system for verification. The system dynamics can be described by the Lagrange method as follows:

$$L = T_1 + T_2 - U \tag{A25}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = T_j \tag{A26}$$

where L is the Lagrange's function. T_1 and T_2 represent the linear and the rotational kinetic energy, respectively. U is the potential energy of the system. q_j represents the generalized coordinates, while T_j, for $j = \theta,\ \psi,\ \phi$ is the applied torque on the corresponding axis. First, T_1, T_2, and U are as follows:

$$T_1 = \frac{1}{2}m\dot{A}_r^2 + \frac{1}{2}m\dot{A}_l^2 + \frac{1}{2}M\dot{B}^2 \tag{A27}$$

$$T_2 = \frac{1}{2}J_w\dot{\theta}_r^2 + \frac{1}{2}J_w\dot{\theta}_l^2 + \frac{1}{2}J_\psi\dot{\psi}^2 + \frac{1}{2}J_\phi\dot{\phi}^2 + \frac{1}{2}J_mn_1^2(\dot{\theta}_r - \dot{\psi})^2 + \frac{1}{2}J_mn_1^2(\dot{\theta}_l - \dot{\psi})^2 \tag{A28}$$

$$U = Mg\ell\cos\psi \tag{A29}$$

Second, T_θ, T_ψ, and T_ϕ can be represented as in the following:

$$T_\theta = \left[\left(2m+M\right)r^2 + 2J_w + 2n_1^2 J_m\right]\ddot{\theta} + \left(Mr\ell\cos\psi - 2n_1^2 J_m\right)\ddot{\psi} - Mr\ell\dot{\psi}^2\sin\psi \tag{A30}$$

$$T_\psi = \left(Mr\ell\cos\psi - 2n_1^2 J_m\right)\ddot{\theta} + \left(M\ell^2 + J_\psi + 2n_1^2 J_m\right)\ddot{\psi} - Mg\ell\sin\psi - M\ell^2\sin\psi\cos\psi\dot{\phi}^2 \tag{A31}$$

$$T_\phi = \left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2 J_m\right) + M\ell^2\sin^2\psi\right]\ddot{\phi} + 2M\ell^2\dot{\psi}\sin\psi\cos\psi\dot{\phi} \tag{A32}$$

Taking linearization about $\psi = 0$ and $\phi = 0$ result in:

$$T_\theta = \left[\left(2m+M\right)r^2 + 2J_w + 2n_1^2 J_m\right]\ddot{\theta} + \left(Mr\ell - 2n_1^2 J_m\right)\ddot{\psi} \tag{A33}$$

$$T_\psi = \left(Mr\ell - 2n_1^2 J_m\right)\ddot{\theta} + \left(M\ell^2 + J_\psi + 2n_1^2 J_m\right)\ddot{\psi} - Mg\ell\psi \tag{A34}$$

$$T_\phi = \left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2 J_m\right)\right]\ddot{\phi} \tag{A35}$$

Note that (A33), (A34), and (A35) are the same as (A19), (A21), and (A24), respectively, considering the following conditions:

$$T_\theta = T_r + T_l,\ T_\psi = -\left(T_r + T_l\right),\ T_\phi = \frac{W}{2r}\left(T_r - T_l\right) \tag{A36}$$

A.4. System Transfer Functions

Taking Laplace transform of (A33-A35) gives the following equations:

$$\hat{T}_\theta = \left[\left(2m+M\right)r^2 + 2J_w + 2n_1^2 J_m\right]s^2\hat{\theta} + \left(Mr\ell - 2n_1^2 J_m\right)s^2\hat{\psi} \tag{A37}$$

$$\hat{T}_\psi = \left(Mr\ell - 2n_1^2 J_m\right)s^2\hat{\theta} + \left(M\ell^2 + J_\psi + 2n_1^2 J_m\right)s^2\hat{\psi} - Mg\ell\hat{\psi} \tag{A38}$$

$$\hat{T}_\phi = \left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2 J_m\right)\right]s^2\hat{\phi} \tag{A39}$$

where $\hat{T}(s) = L\{T(t)\}$ represents the Laplace transform of $T(t)$.

First, considering the translational motion of the TWIP system, we substitute (A5-A7) to the summation of (A37) and (A38). Assume $\theta_m^r = \theta_m^l = \theta_m$ and $\theta_r = \theta_l = \theta$, the transfer function from the pitch angle to the motor angle can be represented as follows:

$$H_c = \frac{\mathcal{L}\{\dot{\theta}_m\}}{\mathcal{L}\{\dot{\psi}\}} = \frac{s\hat{\theta}_m}{s\hat{\psi}} = \frac{-[(2m+M)r^2 + 2J_w + 2Mrl + Ml^2 + J_\psi]s^2 + Mgl}{\left[(2m+M)r^2 + Mrl + 2J_w\right]s^2}$$

which gives equation (1). Then, the relation between the pitch angle ψ and the forward wheel angle θ can be directly derived from (A38), as in the following:

$$\hat{\theta} = \frac{\hat{T}_\psi - [(Ml^2 + J_\psi + 2n_1^2 J_m)s^2 \text{-} Mgl]}{(Mrl - 2n_1^2 J_m)s^2}\hat{\psi} \tag{A40}$$

Substituting (A40) and (A36) into (A37) gives the following relation:

$$G_m = \frac{s\hat{\psi}}{\hat{T}_\psi} = \frac{\left[(2m+M)r^2 + Mrl + 2J_w\right]s}{\left[(2m+M)r^2 + 2J_w + 2n_1^2 J_m\right]\left[\left(Ml^2 + J_\psi + 2n_1^2 J_m\right)s^2 - Mgl\right] - (Mrl - 2n_1^2 J_m)^2 s^2}$$

which is equation (2). Third, substituting the summation of (A37) and (A38) with (A36) gives the following equation:

$$H_v = \frac{rs\hat{\theta}}{\hat{\psi}} = \frac{r\left[-(J_\psi + Ml^2 + Mrl)s^2 + Mgl\right]}{\left[(2m+M)r^2 + Mrl + 2J_w\right]s}$$

which is equation (10). Finally, taking Laplace transform of (A1) and (A2) and applying the above H_c and G_m, we can obtain the system's block diagram, as shown in Figure 2.

Second, considering the steering motion of the TWIP system, we can derive the following equation from (A39):

$$G_t = \frac{s\hat{\phi}}{\hat{T}_\phi} = \frac{1}{\left[\frac{1}{2}mW^2 + J_\phi + \frac{W^2}{2r^2}\left(J_w + n_1^2 J_m\right)\right]s} \tag{A41}$$

Furthermore, we can assume $\theta_l = -\theta_r$, $\theta_m^l = -\theta_m^r$ and rewrite (A8) as

$$\phi = \frac{r}{W}(\theta_r - \theta_l) = \frac{2r}{W}\theta_r = \frac{-2r}{W}\theta_l \tag{A42}$$

Substituting (A5-A7) to the summation of (A37) and (A38), the transfer function from the wheel angle to the motor angle can be represented as follows:

$$H_d = \frac{s\hat{\theta}_m^l}{s\hat{\theta}_l} = \frac{s\hat{\theta}_m^r}{s\hat{\theta}_r} = \frac{[(2m+M)r^2 + 2J_w + 2Mrl + Ml^2 + J_\psi]s^2 - Mgl}{(Mrl + Ml^2 + J_\psi)s^2 - Mgl} \tag{A43}$$

Finally, we can obtain the steering system's block diagram, as shown in Figure A4, from the equations of (A1-A2) and (A41-A43).

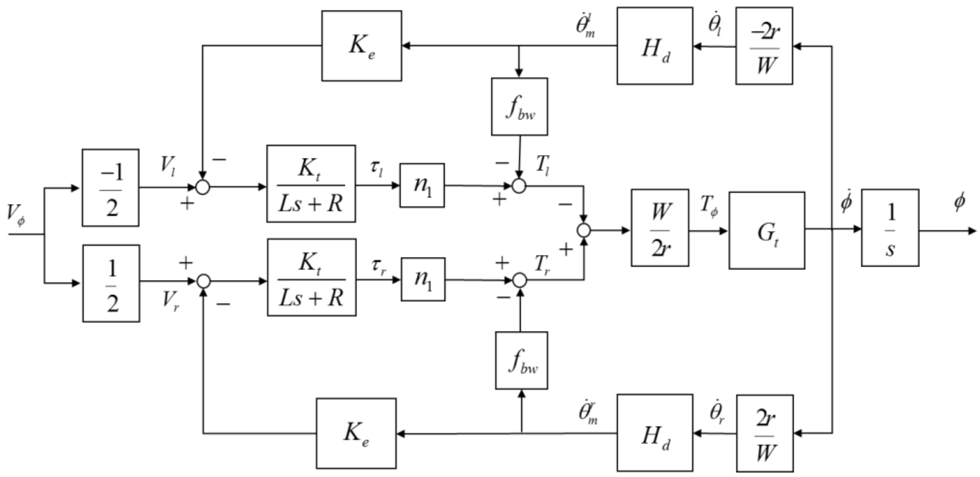

Figure A4. Block diagram of the steering system.

The transfer functions matrix from the motor voltages to the system angles can be expressed as:

$$\begin{bmatrix}\theta \\ \psi \\ \phi\end{bmatrix} = G\begin{bmatrix}V_r \\ V_l\end{bmatrix} = \begin{bmatrix}G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32}\end{bmatrix}\begin{bmatrix}V_r \\ V_l\end{bmatrix} \tag{A44}$$

where V_r and V_l are the right and left motor voltages. Note that $G_{11} = G_{12}$, $G_{21} = G_{22}$, and $G_{31} = -G_{32}$. Substituting the parameters of Table 1 gives

$$G_{11} = G_{12} = \frac{2.963 \times 10^4 s^2 - 7.894 \times 10^{-11} s - 3.003 \times 10^5}{s^5 + 473.8 s^4 + 4402 s^3 - 5994 s^2 - 3.423 \times 10^4 s} \quad \text{(A45)}$$

$$G_{21} = G_{22} = \frac{-9106 s}{s^4 + 473.8 s^3 + 4402 s^2 - 5994 s - 3.423 \times 10^4} \quad \text{(A46)}$$

$$G_{31} = -G_{32} = \frac{2.117 \times 10^4}{s^3 + 474.2 s^2 + 2955 s} \quad \text{(A47)}$$

Suppose the TWIP system moving on the translational direction, i.e.

$$V_r = V_l = \frac{1}{2} V_\psi \quad \text{(A48)}$$

From (A44), the model for the balance control loop part can be described as:

$$\psi = G_{21} V_r + G_{22} V_l = G_{21} V_\psi = \frac{-9106 s}{s^4 + 473.8 s^3 + 4402 s^2 - 5994 s - 3.423 \times 10^4} V_\psi$$

which gives equation (3).