Dynamics of a thirty-one DOF train model

Suppose a thirty-one DOF train travels at a constant speed V on a straight track, the dynamic equations of the model can be derived as follows:

(1) Dynamics of the car body:

$$m_c \ddot{y}_c = F_{syc} \tag{A1}$$

$$m_c \ddot{z}_c = F_{szc} \tag{A2}$$

$$I_{cx} \ddot{\phi}_c = M_{sxc} \tag{A3}$$

$$I_{cy} \ddot{\theta}_c = M_{syc} \tag{A4}$$

$$I_{cz} \ddot{\psi}_c = M_{szc} \tag{A5}$$

where the suspension forces and moments are

$$\begin{aligned} F_{syc} &= 2K_{sy}y_{t1} + 2C_{sy}\dot{y}_{t1} + 2K_{sy}y_{t2} + 2C_{sy}\dot{y}_{t2} - 4K_{sy}y_c \\ &\quad - 4C_{sy}\dot{y}_c - 4K_{sy}h_S\phi_c - 4C_{sy}h_S\dot{\phi}_c \end{aligned} \tag{A6}$$

$$F_{szc} = 2K_{sz}z_{t1} + 2C_{sz}\dot{z}_{t1} + 2K_{sz}z_{t2} + 2C_{sz}\dot{z}_{t2} - 4K_{sz}z_c - 4C_{sz}\dot{z}_c \tag{A7}$$

$$\begin{aligned} M_{sxc} &= 2K_{sy}h_S(y_{t1} + y_{t2}) + 2C_{sy}h_S(\dot{y}_{t1} + \dot{y}_{t2}) + 2b_2^2K_{sz}(\phi_{t1} + \phi_{t2}) \\ &\quad + 2b_2^2\mathrm{C}_{sz}(\dot{\phi}_{t1} + \dot{\phi}_{t2}) - 4K_{sz}b_2^2\phi_c - 4C_{sz}b_2^2\dot{\phi}_c - 4K_{sy}h_S^2\phi_c \\ &\quad - 4C_{sy}h_S^2\dot{\phi}_c - 4h_S(K_{sy}y_c + C_{sy}\dot{y}_c) \end{aligned} \tag{A8}$$

$$\begin{aligned} M_{syc} &= 2K_{sz}L_2(-z_{t1} + z_{t2}) + 2C_{sz}L_2(-\dot{z}_{t1} + \dot{z}_{t2}) \\ &\quad - (4K_{sz}L_2^2 + 4K_{sx}h_S^2)\theta_c - (4C_{sz}L_2^2 + 4C_{sx}h_S^2)\dot{\theta}_c \end{aligned} \tag{A9}$$

$$\begin{aligned} M_{szc} &= 2K_{sy}L_2(y_{t1} - y_{t2}) + 2C_{sy}L_2(\dot{y}_{t1} - \dot{y}_{t2}) \\ &\quad + 2K_{sx}b_2^2(\psi_{t1} + \psi_{t2}) + 2C_{sx}b_2^2(\dot{\psi}_{t1} + \dot{\psi}_{t2}) \\ &\quad - 4L_2^2(K_{sy}\psi_c + C_{sy}\dot{\psi}_c) - 4b_2^2(K_{sx}\psi_c + C_{sx}\dot{\psi}_c) \end{aligned} \tag{A10}$$

(2) Dynamics of the bogies: (i=1 and 2 for the front and rear bogies, respectively. And i=1 when j=1 or 2; i=2 when j=3 or 4)

$$m_t \ddot{y}_{ti} = F_{syt} \tag{A11}$$

$$m_t \ddot{z}_{ti} = F_{szt} \tag{A12}$$

$$I_{tx} \ddot{\phi}_{ti} = M_{sxt} \tag{A13}$$

$$I_{ty} \ddot{\theta}_{ti} = M_{syt} \tag{A14}$$

$$I_{tz} \ddot{\psi}_{ti} = M_{szt} \tag{A15}$$

where the suspension forces and moments are

$$\begin{aligned} F_{syti} = & 2K_{py}(y_j + y_{j+1}) + 2C_{py}(\dot{y}_j + \dot{y}_{j+1}) - (4K_{py} + 2K_{sy})y_{ti} \\ & - (4C_{py} + 2C_{sy})\dot{y}_{ti} - 4K_{py}h_T\phi_{ti} - 4C_{py}h_T\dot{\phi}_{ti} \\ & + 2(K_{sy}y_c + C_{sy}\dot{y}_c) + 2h_S(K_{sy}\phi_c + C_{sy}\dot{\phi}_c) \\ & + 2(-1)^{i+1}L_2(K_{sy}\psi_c + C_{sy}\dot{\psi}_c) \end{aligned} \tag{A16}$$

$$\begin{aligned} F_{szt} = & -2K_{sz}z_{ti} - 2C_{sz}\dot{z}_{ti} - 4K_{pz}z_{ti} - 4C_{pz}\dot{z}_{ti} + 2K_{sz}z_c + 2C_{sz}\dot{z}_c \\ & + 2K_{pz}(z_j + z_{j+1}) + 2C_{pz}(\dot{z}_j + \dot{z}_{j+1}) \\ & + (-1)^i 2L_2K_{sz}\theta_c + (-1)^i 2L_2C_{sz}\dot{\theta}_c \end{aligned} \tag{A17}$$

$$\begin{aligned} M_{sxti} = & 2K_{pz}b_1^2(\phi_j + \phi_{j+1}) + 2C_{pz}b_1^2(\dot{\phi}_j + \dot{\phi}_{j+1}) \\ & + 2K_{py}h_T(y_j + y_{j+1}) + 2C_{py}h_T(\dot{y}_j + \dot{y}_{j+1}) \\ & - 2b_2^2(K_{sz}\phi_{ti} + C_{sz}\dot{\phi}_{ti}) - 4b_1^2(K_{pz}\phi_{ti} + C_{pz}\dot{\phi}_{ti}) \\ & - 4h_T^2(K_{py}\phi_{ti} + C_{py}\dot{\phi}_{ti}) - 4h_T(K_{py}y_{ti} + C_{py}\dot{y}_{ti}) \\ & + 2b_2^2(K_{sz}\phi_c + C_{sz}\dot{\phi}_c) \end{aligned} \tag{A18}$$

$$\begin{aligned} M_{syti} = & -4h_T^2(K_{px}\theta_{ti} + C_{px}\dot{\theta}_{ti}) - 4L_1^2(K_{pz}\theta_{ti} + C_{pz}\dot{\theta}_{ti}) \\ & + 2L_1(-K_{pz}z_j - C_{pz}\dot{z}_j + K_{pz}z_{j+1} + C_{pz}\dot{z}_{j+1}) \end{aligned} \tag{A19}$$

$$\begin{aligned} M_{szti} = & 2b_2^2(K_{sx}\psi_c + C_{sx}\dot{\psi}_c) - (4K_{py}L_1^2 + 4K_{px}b_1^2 + 2K_{sx}b_2^2)\psi_{ti} \\ & - (4C_{py}L_1^2 + 4C_{px}b_1^2 + 2C_{sx}b_2^2)\dot{\psi}_{ti} + 2K_{py}L_1(y_j - y_{j+1}) \\ & + 2C_{py}L_1(\dot{y}_j - \dot{y}_{j+1}) + 2K_{px}b_1^2(\psi_j + \psi_{j+1}) + 2C_{px}b_1^2(\dot{\psi}_j + \dot{\psi}_{j+1}) \end{aligned} \tag{A20}$$

(3) Dynamics of the wheel-sets: (j = 1–4 for the four wheel-sets from front to rear respectively, and i=1 when j=1or 2; i=2 when j=3 or 4)

$$m_w \ddot{y}_j = F_{Lyj} + F_{Ryj} + N_{Lyj} + N_{Ryj} + F_{syj} \tag{A21}$$

$$m_w \ddot{z}_j = F_{Lzj} + F_{Rzj} + N_{Lzj} + N_{Rzj} + F_{szj} - W \tag{A22}$$

$$\begin{aligned} I_{wx} \ddot{\phi}_j = I_{wy} \frac{V}{r_0} \dot{\psi}_j &+ (R_{Ryj} F_{Rzj} - R_{Rzj} F_{Ryj}) + (R_{Lyj} F_{Lzj} - R_{Lzj} F_{Lyj}) \\ &+ (R_{Ryj} N_{Rzi} - R_{Rzj} N_{Ryj}) + (R_{Lyj} N_{Lzj} - R_{Lzj} N_{Lyj}) \\ &+ M_{Lxj} + M_{Rxj} + M_{sxj} \end{aligned} \tag{A23}$$

$$\begin{aligned} I_{wz} \ddot{\psi}_j = -I_{wy} \frac{V}{r_0} \dot{\phi}_j &+ (R_{Rxj} F_{Ryj} - R_{Ryj} F_{Rxj}) + (R_{Lxj} F_{Lyj} - R_{Lyj} F_{Lxj}) \\ &+ (R_{Rxj} N_{Ryj} + R_{Lxj} N_{Lyj}) + M_{Lzj} + M_{Rzj} + M_{szj} \end{aligned} \tag{A24}$$

in which the suspension forces and moments are

$$\begin{aligned} F_{syj} = &-2K_{py} y_j - (-1)^j 2K_{py} L_1 \psi_{ti} + 2K_{py} y_{ti} \\ &- 2C_{py} y_j - (-1)^j 2C_{py} L_1 \psi_{ti} + 2C_{py} \dot{y}_{ti} \\ &+ 2K_{py} \phi_{ti} h_T + 2C_{py} \dot{\phi}_{ti} h_T \end{aligned} \tag{A25}$$

$$\begin{aligned} F_{szj} = &-2K_{pz} z_j - 2C_{pz} \dot{z}_j + 2K_{pz} z_{ti} + 2C_{pz} \dot{z}_{ti} \\ &+ (-1)^j 2L_1 (K_{pz} \theta_{ti} + C_{pz} \dot{\theta}_{ti}) \end{aligned} \tag{A26}$$

$$M_{sxj} = -2K_{pz} b_1^2 \phi_j - 2C_{pz} b_1^2 \dot{\phi}_j + 2K_{pz} b_1^2 \phi_{ti} + 2C_{pz} b_1^2 \dot{\phi}_{ti} \tag{A27}$$

$$M_{szj} = 2K_{px} b_1^2 \psi_{ti} - 2K_{px} b_1^2 \psi_j + 2C_{px} b_1^2 \dot{\psi}_{ti} - 2C_{px} b_1^2 \dot{\psi}_j \tag{A28}$$

The rail interactions, which include the creep forces, creep moments, and normal forces, can be expressed as in the following:

(4) Creep forces on the left wheel:

$$F_{Lxj} = F_{Lxj}^* - F_{Lyj}^* \psi_j \tag{A29}$$

$$F_{Lyj} = F_{Lxj}^* \psi_j + F_{Lyj}^* \tag{A30}$$

$$F_{Lzj} = F_{Lyj}^* (\delta_L + \phi_j) \tag{A31}$$

(5) Creep forces on the right wheel:

$$F_{Rxj} = F^*_{Rxj} - F^*_{Ryj}\psi_j \tag{A32}$$

$$F_{Ryj} = F^*_{Rxj}\psi_j + F^*_{Ryj} \tag{A33}$$

$$F_{Rzj} = -F^*_{Ryj}(\delta_R - \phi_j) \tag{A34}$$

(6) Creep moments on the left wheel:

$$M_{Lxj} = M^*_{Lzj}(\delta_L + \phi_j)\psi_j \tag{A35}$$

$$M_{Lzj} = M^*_{Lzj} \tag{A36}$$

(7) Creep moments on the right wheel:

$$M_{Rxj} = -M^*_{Rzj}(\delta_R - \phi_j)\psi_j \tag{A37}$$

$$M_{Rzj} = M^*_{Rzj} \tag{A38}$$

In equations (A29)–(A38), according to Kaller theory [8], $F^*_{Lxj}, F^*_{Lyj}, M^*_{Lzj}, F^*_{Rxj}, F^*_{Ryj}$ and M^*_{Rzj} are defined as:

$$F^*_{Lxj} = -\frac{f_{33}}{V}\left[V\left(1-\frac{r_L}{r_0}\right) - a\dot{\psi}_j\right] \tag{A39}$$

$$F^*_{Lyj} = -\frac{f_{11}}{V}\left[\dot{y}_j + r_{Lj}\dot{\phi}_j - V\psi_j\right] - \frac{f_{12}}{V}\left[\dot{\psi}_j - \frac{V}{r_0}\delta_L\right] \tag{A40}$$

$$M^*_{Lzj} = \frac{f_{12}}{V}\left[\dot{y}_j + r_{Lj}\dot{\phi}_j - V\psi_j\right] - \frac{f_{22}}{V}\left[\dot{\psi}_j - \frac{V}{r_0}\delta_L\right] \tag{A41}$$

$$F^*_{Rxj} = -\frac{f_{33}}{V}\left[V\left(1-\frac{r_{Rj}}{r_0}\right) + a\dot{\psi}_j\right] \tag{A42}$$

$$F_{Ryj}^{*} = -\frac{f_{11}}{V}\left[\dot{y}_j + r_{Rj}\dot{\phi}_j - V\psi_j\right] - \frac{f_{12}}{V}\left[\dot{\psi}_j + \frac{V}{r_0}\delta_R\right] \tag{A43}$$

$$M_{Rzj}^{*} = \frac{f_{12}}{V}\left[\dot{y}_j + r_{Rj}\dot{\phi}_j - V\psi_j\right] - \frac{f_{22}}{V}\left[\dot{\psi}_j + \frac{V}{r_0}\delta_L\right] \tag{A44}$$

(8) The normal forces between the wheel-sets and the rail:

$$N_{Lzj} = N_{Rzj} = \frac{1}{2}W \tag{A45}$$

$$N_{Lyj} = -N_{Lzj}\tan(\delta_L + \phi_j) \approx -\frac{1}{2}W(\delta_L + \phi_j) \tag{A46}$$

$$N_{Ryj} = N_{Rzj}\tan(\delta_R - \phi_j) \approx \frac{1}{2}W(\delta_R - \phi_j) \tag{A47}$$

In addition, the position vectors of the contact point between wheels and the rail can be expressed as follows:

$$R_{Rxj} = a\psi_j \tag{A48}$$

$$R_{Ryj} = -a + r_{Rj}\phi_j \tag{A49}$$

$$R_{Rzj} = -a\phi_j - r_{Rj} \tag{A50}$$

$$R_{Lxj} = -a\psi_j \tag{A51}$$

$$R_{Lyj} = a + r_{Lj}\phi_j \tag{A52}$$

$$R_{Lzj} = a\phi_j - r_{Lj} \tag{A53}$$

where δ_L, δ_R, r_{Li} and r_{Ri} can be derived by the following geometric relations:

$$\delta_L = \delta_R = \lambda \ , \ \frac{1}{2}(r_{Lj} - r_{Rj}) = \lambda y_j \ , \ \frac{1}{2}(r_{Lj} + r_{Rj}) = r_0 \tag{A54}$$

Substituting equations (A25)–(A54) into equations (A21)–(A24), the dynamic of the wheel-sets can be obtained as follows:

$$m_w \ddot{y}_j = -\frac{2f_{11}}{V}\dot{y}_j - W\phi_j - \frac{2r_0 f_{11}}{V}\dot{\phi}_j + 2f_{11}\psi_j - \frac{2f_{12}}{V}\dot{\psi}_j + F_{syj} \tag{A55}$$

$$\begin{aligned} m_w \ddot{z}_j = &-\frac{2f_{11}\lambda^2}{V} y_j\dot{\phi}_j - \frac{2f_{11}}{V}\dot{y}_j\phi_j - \frac{2r_0 f_{11}}{V}\phi_j\dot{\phi}_j \\ &-\frac{2f_{12}}{V}\phi_j\dot{\psi}_j + \frac{2f_{12}\lambda^2}{r_0} + F_{szj} \end{aligned} \tag{A56}$$

$$\begin{aligned} I_{wx}\ddot{\phi}_j = &\left(\frac{2f_{12}\lambda^2}{r_0} - \lambda^2 W\right) y_j - \frac{2f_{11}(r_0 + a\lambda)}{V}\dot{y}_j + (2f_{12}\lambda^2 + a\lambda W)\phi_j \\ &-\left(\frac{2f_{11}ar_0\lambda}{V} + \frac{2f_{11}r_0^2}{V}\right)\dot{\phi}_j + \left[2f_{11}(r_0 + a\lambda) + \frac{2f_{22}\lambda^2}{r_0}\right]\psi_j \\ &+\left(\frac{I_{wy}V}{r_0} - \frac{2f_{12}r_0}{V} - \frac{2f_{12}a\lambda}{V}\right)\dot{\psi}_j + M_{sxj} \end{aligned} \tag{A57}$$

$$\begin{aligned} I_{wz}\ddot{\psi}_j = &-\frac{2af_{33}\lambda}{r_0} y_j + \frac{2f_{12}}{V}\dot{y}_j + \left(-\frac{I_{wy}V}{r_0} + \frac{2f_{12}r_0}{V}\right)\dot{\phi}_j \\ &+(-2f_{12} + a\lambda W)\psi_j - \left(\frac{2a^2 f_{33}}{V} + \frac{2f_{22}}{V}\right)\dot{\psi}_j + M_{szj} \end{aligned} \tag{A58}$$

The dynamic model of the thirty-one DOF full-train model can be derived by equations (A1)–(A5), (A11)–(A15) and (A55)–(A58).