



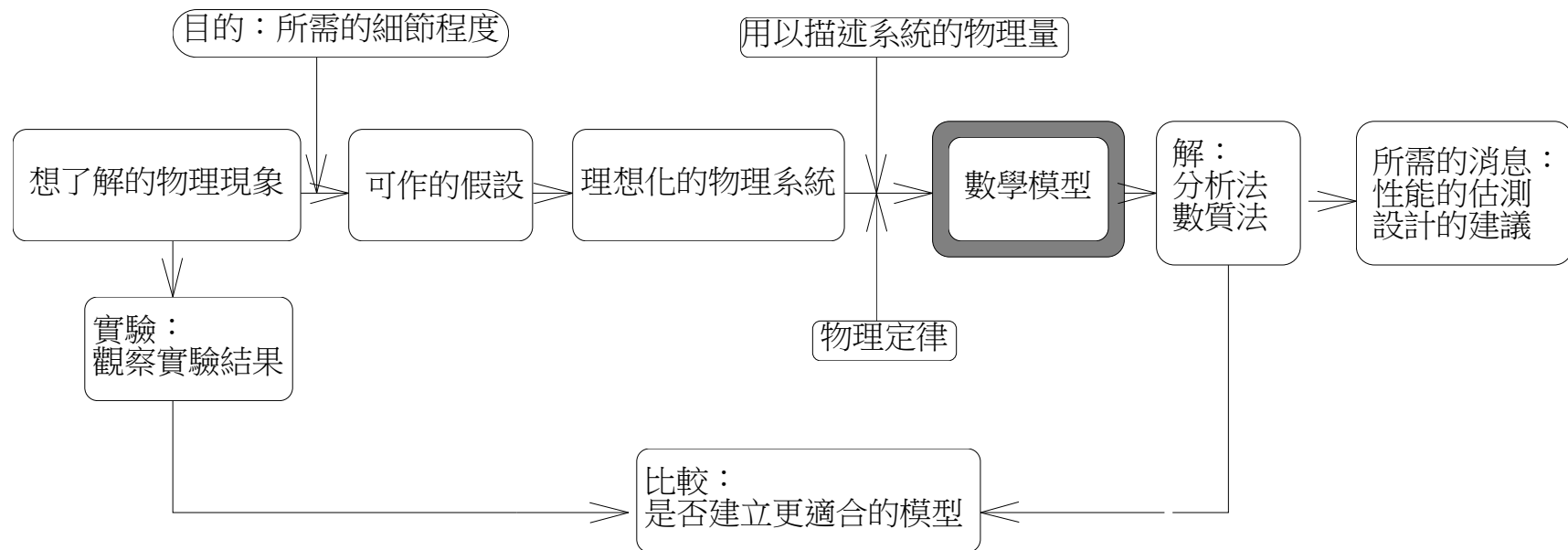
# System Dynamics

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# 系統動態分析簡介

- 靜態系統 (Static systems): 用以描述系統的物理量不隨時間改變的系統
- 動態系統 (Dynamic systems): 用來描述系統的物理量可隨時間改變的系統





## 物理定律與數學表達

- 牛頓定律
- 質量不減定律
- 熱力學定律
- 電磁定律 (Maxwell equations)
- ...

數學模型的目的：

- 了解對象系統的特性
- 預測真實系統在各種參數變化下的行為
- 替代昂貴的實驗
- 系統鑒別 (system identification) 未知參數



# 機械系統

線性動作：

位移：

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{v} = d\mathbf{r} / dt = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

$$\mathbf{a} = d\mathbf{v} / dt = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$

$$\mathbf{p} = m\mathbf{v}$$

動作：

$$m\ddot{x} = F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

$$m\ddot{y} = F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

$$m\ddot{z} = F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$$

二度空間極座標：

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

Equations of motion

$$m(\ddot{r} - r\dot{\theta}^2) = F_r(r, \theta, \dot{r}, \dot{\theta}, t)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta(r, \theta, \dot{r}, \dot{\theta}, t)$$

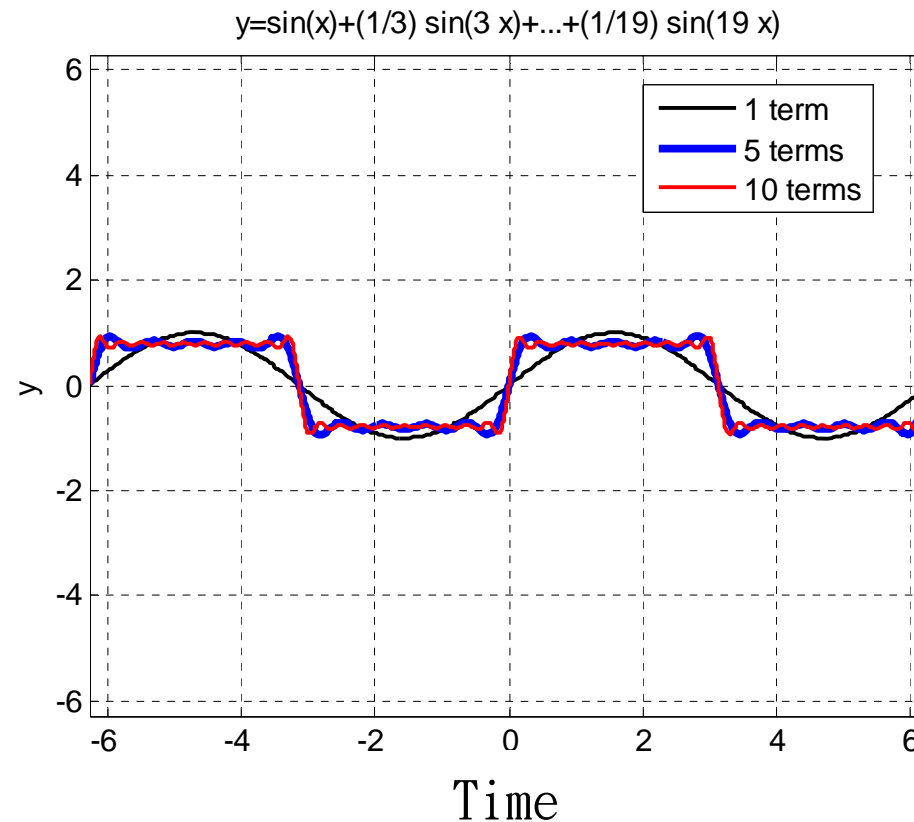


# 基本線性系統表示法

考慮一個時域(time-domain)信號：

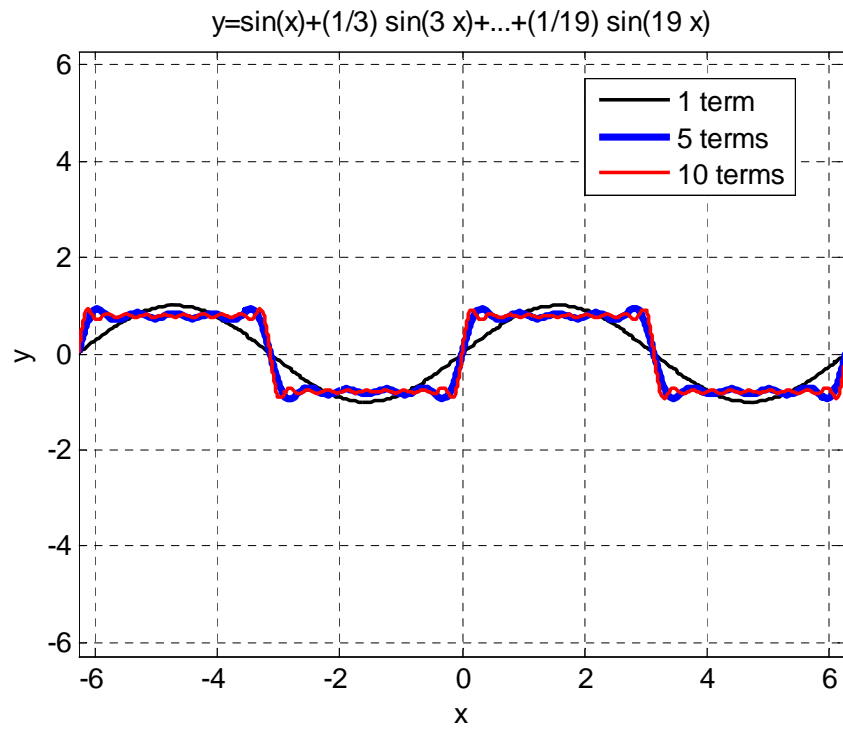
$$f(t) = \sin(t) + (1/3)\sin(3t) + (1/5)\sin(5t) + \dots$$

這些sin函數相加  
成一個方形函數  
 $f(t)$ ，或說 $f(t)$ 含有  
 $\sin(t)$ ，  
 $1/3\sin(3t)$ ， $\dots$ 的  
因子，所有的信號  
都可以用sin與cos  
函數合成出來



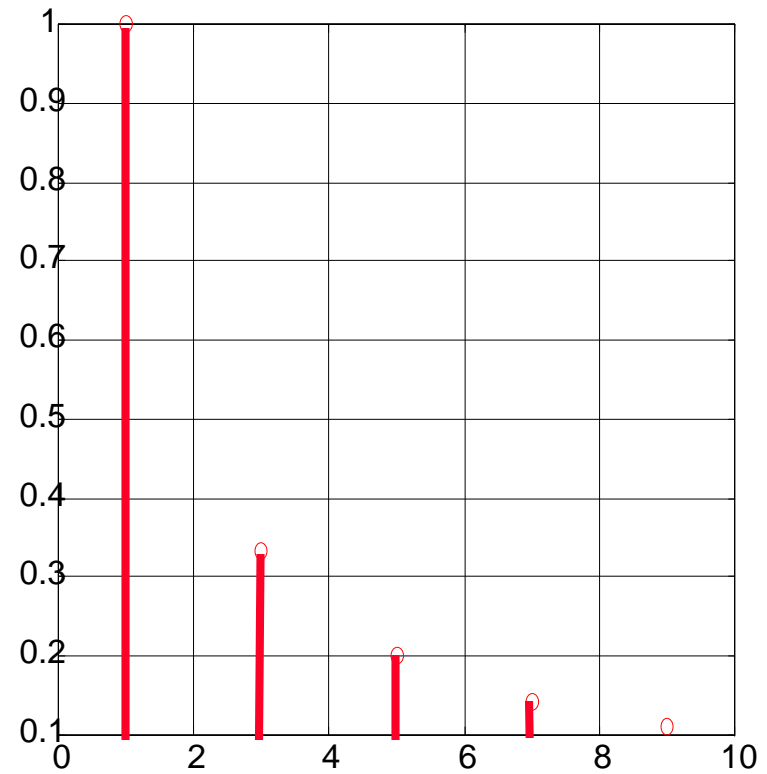


## Time domain



Time

## Frequency domain



Frequency



# Fourier Transform

對一個single-valued function  $f(t)$ ,  
其Fourier Transform 定義為

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \mathcal{F}[f(t)]$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega = \mathcal{F}^{-1}[F(\omega)]$$



(1768-1830)

- Fourier Transform 將信號中每一個”頻率”的成份分析出來。
- 一個信號可以表示在 time domain,  $f(t)$ , 及 frequency domain,  $F(\omega)$
- $\mathcal{F}^{-1}$  稱為 Fourier function 的反函數
- $F(\omega)$  存在的條件：
$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$



# Laplace Transform

當 $f(t)$  不收斂為零以致

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

不成立時，為了放鬆這個條件，我們乘上 $e^{-\sigma t}$ ，只要是

$$\int_{-\infty}^{\infty} |f(t)| e^{-\sigma t} dt < \infty$$

就可以運算，這個轉換定義成

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \mathcal{L}[f(t)] \quad s = \sigma + j\omega$$

Laplace transform 的反運算

$$f(t) = \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s) e^{st} dt = \mathcal{L}^{-1}[F(s)] \quad \sigma_1 > \sigma$$



(1749-1827)





# 線性系統 (Linear Systems)

考慮系統

$$y_u = L(u)$$

所謂線性系統的定義：

1. 加成性 (principle of superposition)

$$y_{u_1+u_2} = L(u_1 + u_2) = L(u_1) + L(u_2) = y_{u_1} + y_{u_2}$$

2. 同次性 (homogeneity)

$$y_{\beta u} = L(\beta u) = \beta L(u) = \beta y_u$$



## 迴旋與響應 (Response by Convolution)

**Principle of superposition** - if the system has an input that can be expressed as a sum of signals, then the response of the system can be expressed as the same sum of the individual responses to the respective signals.

脈衝信號 (Impulse signal):  $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$

脈衝響應 (Impulse Response)

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$$

$u(t)$  can be represented as the sum of impulses of intensity  $u(t-\tau)$ , thus

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$$



## 迴旋與響應 (Response by Convolution)

For a general linear system, we can express the **impulse response** as  $h(t, \tau)$ ,

From principle of superposition, we have the total response of  $u(t)$  is

**Superposition integral:** 
$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t, \tau)d\tau$$

If the system is also constant, then the impulse is given by  $h(t-\tau)$ , the response becomes

**Convolution integral:** 
$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau$$



# 轉移函數 (Transfer Function)

考慮線性非時變系統

$$\frac{dx(t)}{dt} = -\tau x(t) + \tau u(t)$$

取Laplace Transform

$$sX(s) - x_0 = -\tau X(s) + \tau U(s) \quad \text{所以} \quad X(s) = \frac{\tau}{s + \tau} U(s) + \frac{\tau}{s + \tau} x_0$$

考慮起始狀態為零的情形

$$\frac{X(s)}{U(s)} = \frac{\tau}{s + \tau}$$

$$\frac{Y(s)}{U(s)} = G(s)$$

轉移函數  $G(s)$  :

輸出的Laplace transform 對輸入的Laplace transform 的比值



# 頻率響應

## 波德圖 (Bode Plot)

考慮轉移函數

$$Y(s) = G(s)U(s)$$

若將  $s = j\omega$  由於 Laplace transform 中  $s = \sigma + j\omega$  的定義可以得到系統的頻率響應

$$Y(j\omega) = G(j\omega)U(j\omega)$$

$G(j\omega)$  是一個對  $\omega$  的複數函數

$$\begin{aligned} G(j\omega) &= G_R(j\omega) + jG_I(j\omega) \\ &= |G(j\omega)|\angle G(j\omega) \end{aligned}$$

把  $|G(j\omega)|$  和  $\angle G(j\omega)$  沿著頻率軸畫出來就是一種頻率響應 Bode plot

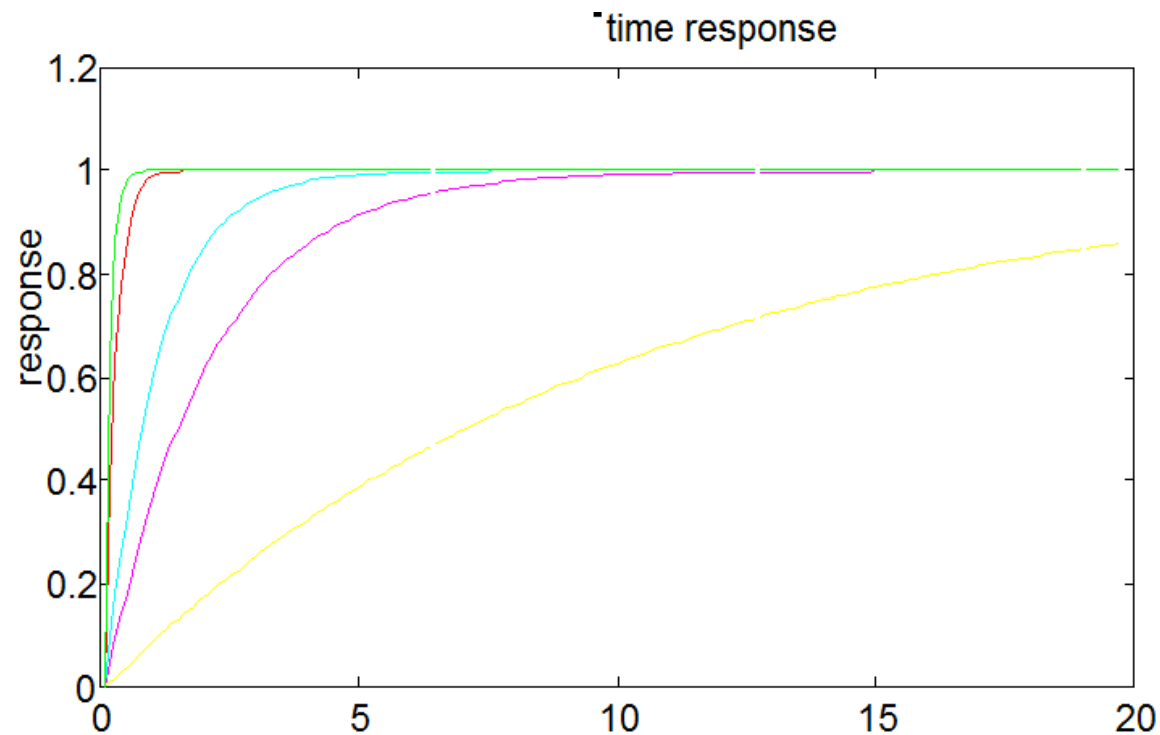


# 動態系統特性

## 一階系統 (first-order system)

其響應為  $x(t) = 1 - e^{-t}$

時間常數之影響：時間常數  $\tau = 0.1, 0.5, 1, 5, 10$





# 動態系統特性

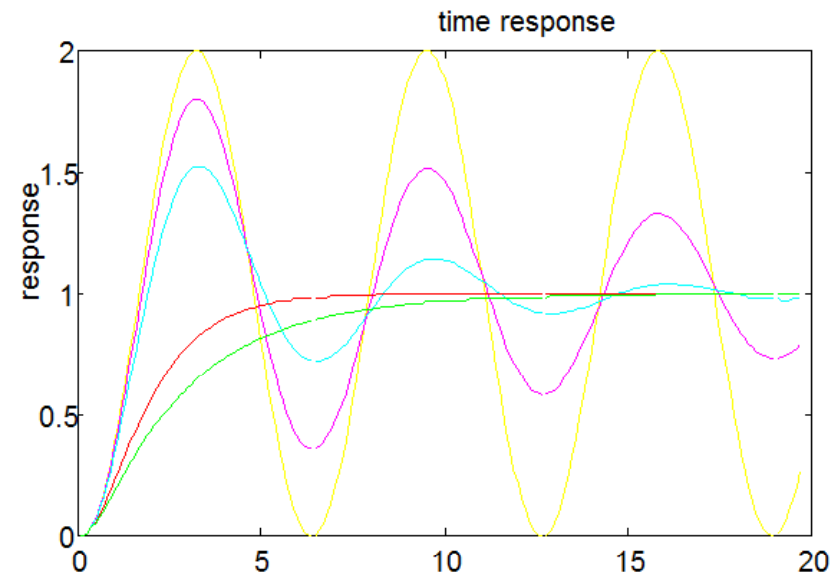
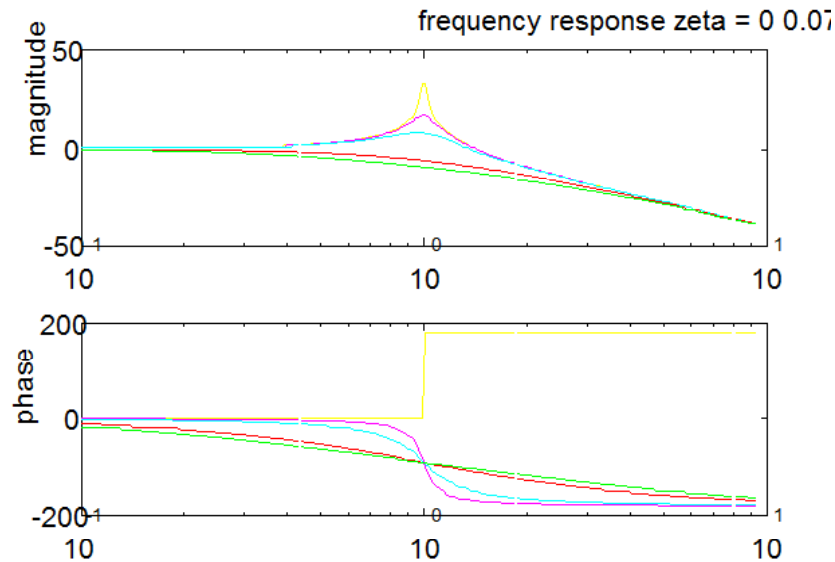
## 二階系統 (second-order system)

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\sqrt{1-\zeta^2}\omega_n t - \phi_0)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

阻尼常數 = 0, 0.07, 0.2, 1.5



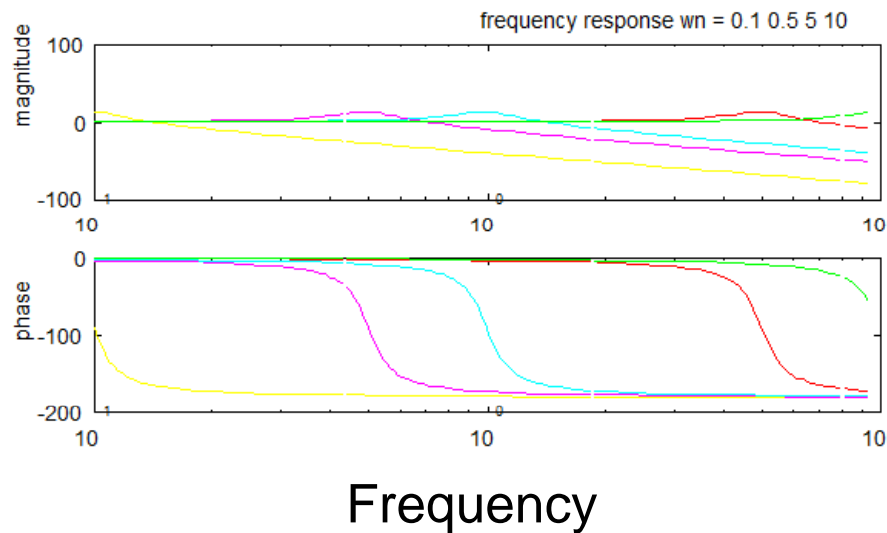


# 動態系統特性

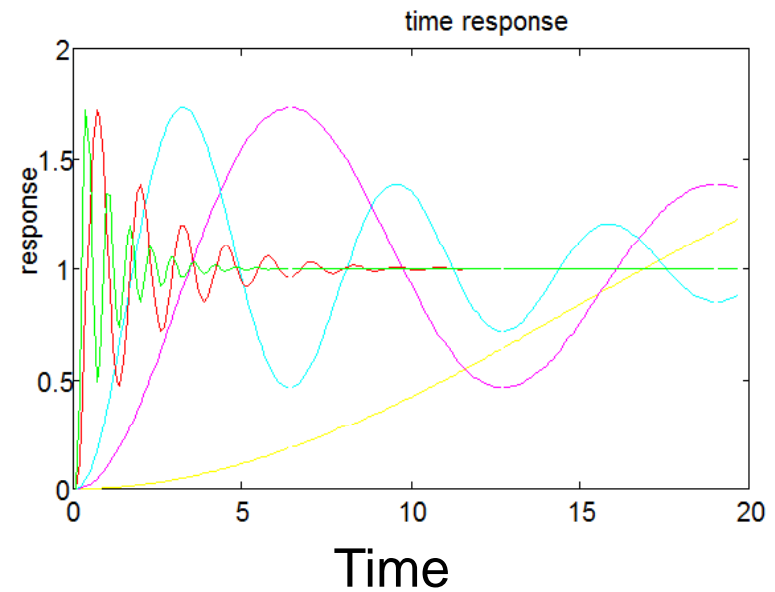
## 二階系統 (second-order system)

自然頻率 = 0.1, 0.5, 1, 5, 10

### Frequency response



### Time response

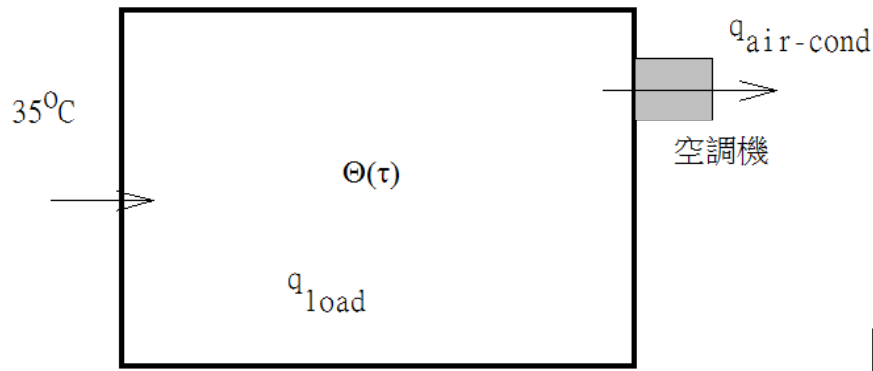






# 簡易空調系統模擬

考慮一約五坪大房間



冷氣開動時，其數學模型為：

$$c_s \frac{d\Theta(\tau)}{d\tau} = q_{load} - q_{air-cond} = \Delta q_{load} \longrightarrow \frac{\Theta(s)}{\Delta Q(s)} = \frac{1}{c_s s}$$

冷氣未開時，其數學模型為：

$$c_s \frac{d\Theta(\tau)}{d\tau} = q_{load} \longrightarrow \frac{\Theta(s)}{\Theta_{load}(s)} = \frac{1}{c_s s}$$



## 簡易空調系統模擬 (cont.)

此時我們有興趣的系統受兩個方程式所統御  
五坪大房間的熱含量約

$$3.306(m^2) \times 4(m) \times 5 \times 1.1774(kg / m^3) \times 1.0057(kJ / kg \cdot ^\circ C) \\ = 78.292(kJ / ^\circ C)$$

$C_s$  約為78.29 KJ/°C

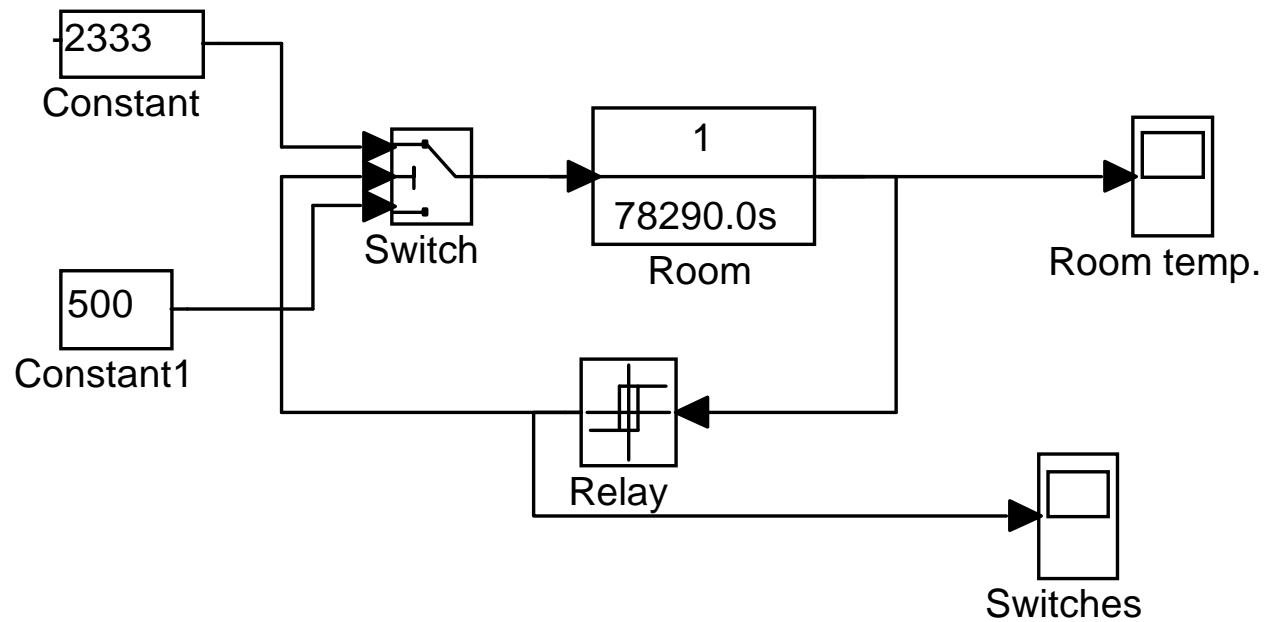
冷氣機之冷凍能力  $q_{\text{air-cond}}$  約2000 kcal/hr = 2.333 KJ/s  
假設房間的熱產生量為 500 KJ/s，所以房間的模型為

$$\frac{\Theta(s)}{Q(s)} = \frac{1}{78290s}$$



## 簡易空調系統模擬 (cont.)

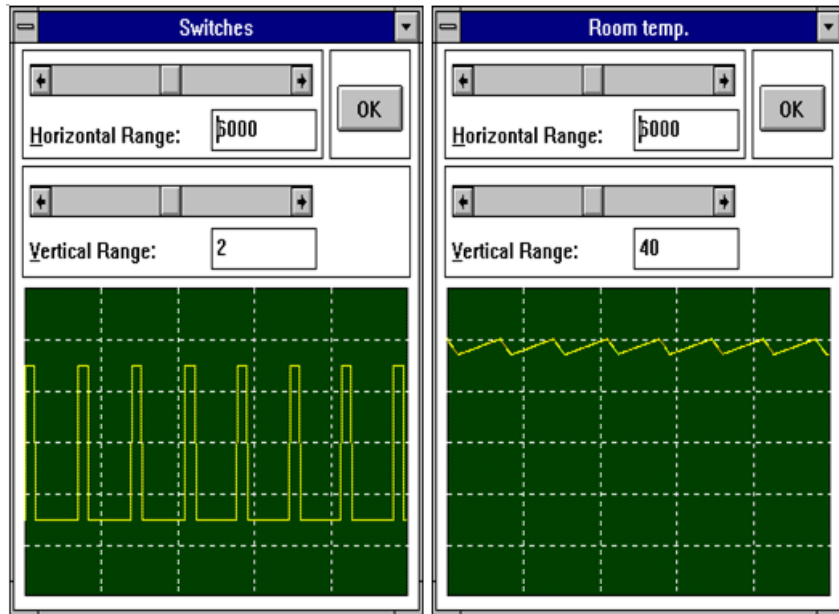
其控制架構如圖：



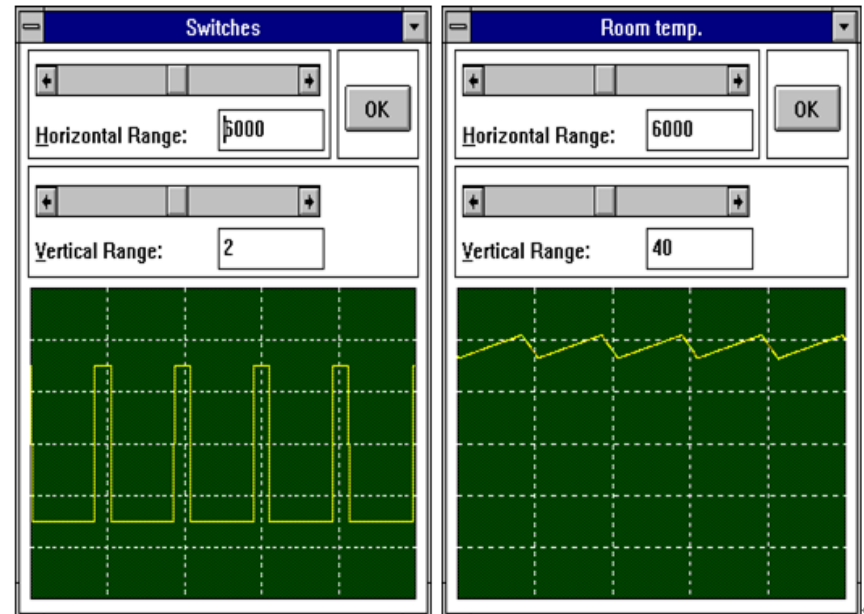


## 簡易空調系統模擬 (cont.)

若以27度與23度作為開關的設定值，  
其控制效果及動作如下：



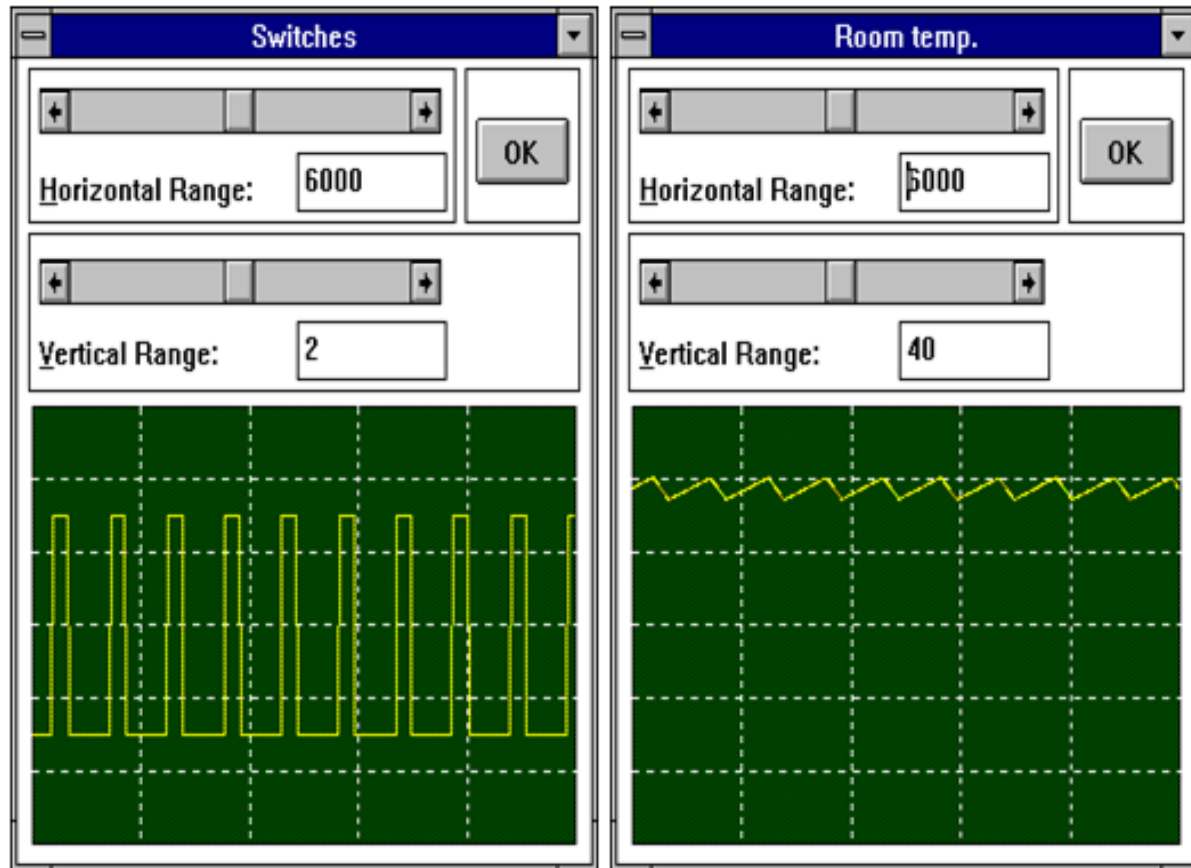
若以28度與22度作為開關的設定值，  
其控制效果及動作如下：





## 簡易空調系統模擬 (cont.)

溫度的變化範圍加大，但機器開關次數減少了。  
考慮負荷加大 700 KJ/s





## 系統動態分析基本理論

從數學的角度而言，所有的系統都可以最後用常微分方程式 (ordinary Differential Equation, ODE) 來表示。做系統的分析可以先假設我們手上有一組足以描述”系統行為”的ODE模型。

- 足夠的”狀態變數”
- 足夠的資料”解出” 隨時間的變化

假設系統模型為

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, t) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, t) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, t) \end{aligned} \quad \text{或} \quad \begin{aligned} \dot{x}_i &= f_i(x_1, x_2, \dots, x_n, t) \\ i &= 1, 2, \dots, n \end{aligned}$$



## 系統動態分析基本理論

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n, t), \quad i = 1, 2, \dots, n$$

其中  $t$  是 independent variable,  $n$  個”狀態”  $x_1, x_2, \dots, x_n$  是 dependent variables.

系統如果不是分開來 (explicitly) 隨時間  $t$  變化稱之為 Autonomous system

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n$$

$f_i(x_1, x_2, \dots, x_n, t)$  可以是  $x_1(t)$  的線性或非線性函數



## 相位(狀態)空間 State Space

假設我們有方法積分或是解出 $x_i(t)$  來  
將這些狀態寫成一個”向量” 的形式，得到一個”狀態向量”

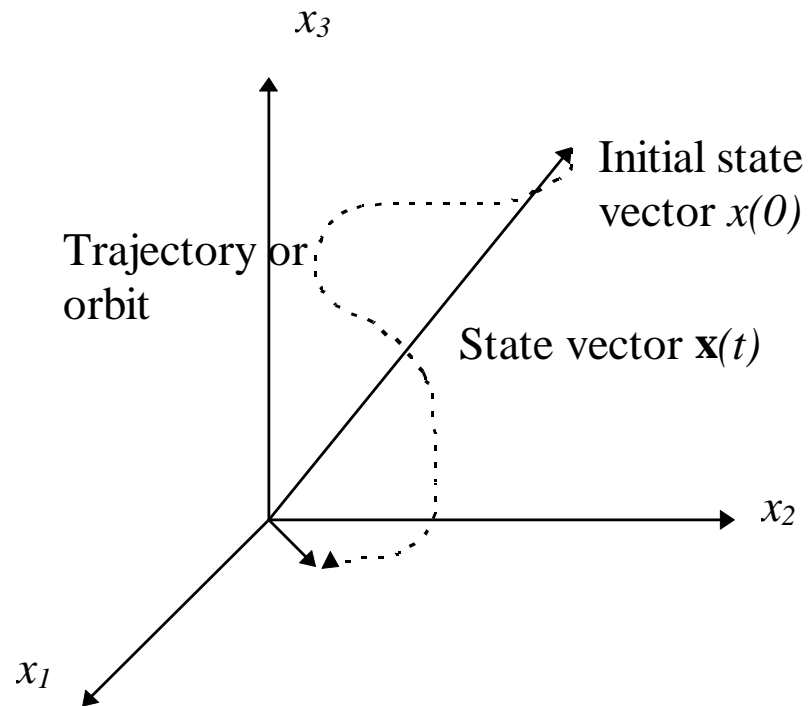
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

這個向量所存在的空間稱為”狀態空間” (State space, phase space)





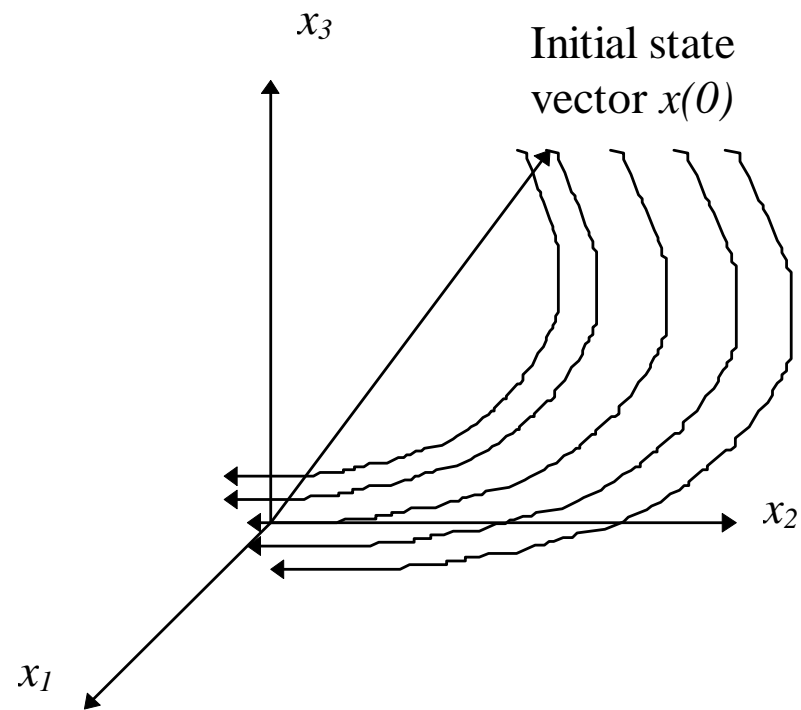
$\mathbf{x}(t)$  隨時間變化在空間中所掃出來的”軌跡” 稱為”系統響應軌跡” (response trajectory) 或”狀態軌跡” (state trajectory)



如果我們選取一堆起始點，這些點會掃出一束軌跡來，稱為相流(phase flow)，可與流力中的stream line 相比



Phase flow 是由狀態方程式所決定，藉著一個系統的相流，可以知道系統的動態特性





# Example 1

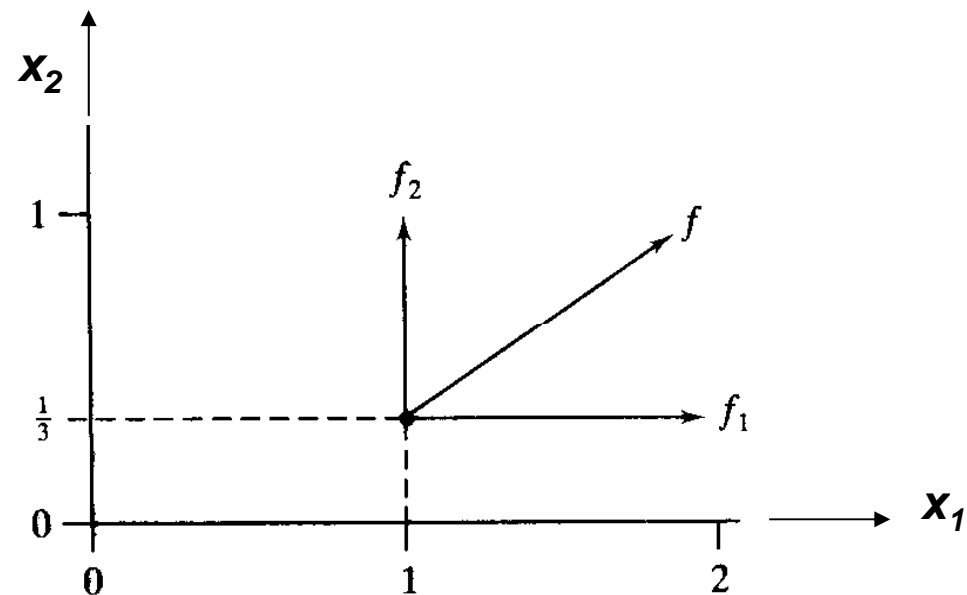
系統

$$f_1(x_1, x_2) = \dot{x}_1 = x_1$$

$$f_2(x_1, x_2) = \dot{x}_2 = x_1^2 - x_2$$

每一點代進其  $f_1$  與  $f_2$  函數值，比方  $(1, 1/3)$  點得

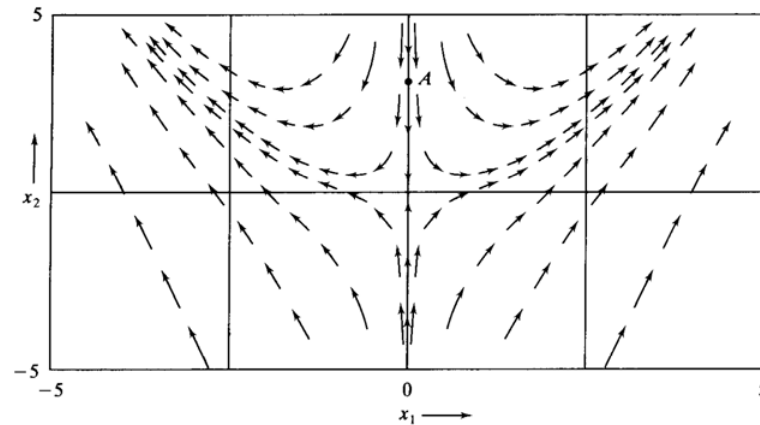
$$f_1 = 1, \quad f_2 = \frac{2}{3}$$



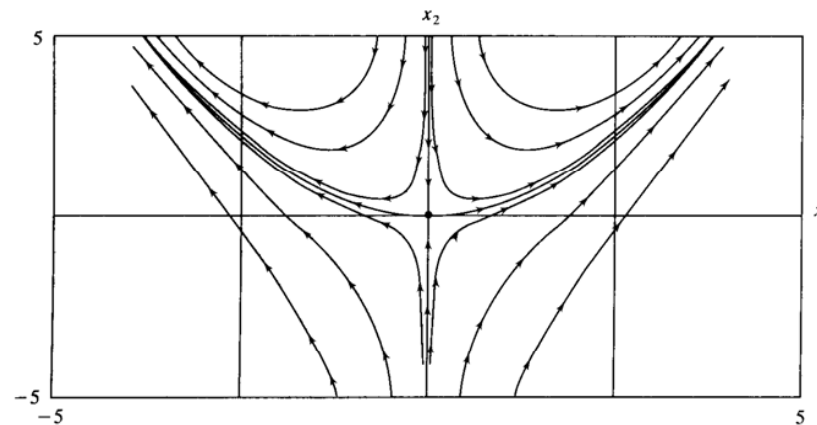


## Example 1 (cont.)

將很多點的向量按"方向"、"大小" 畫出來



這個系統的相流圖可以將這些向量連起來得到





## 平衡點 (equilibrium)

$\dot{\mathbf{x}} = 0$  的點，在這一點系統靜止不動 ( $f(\mathbf{x})=0$ )

$$f_1(x_1, x_2, \dots, x_n) = 0$$

$$f_2(x_1, x_2, \dots, x_n) = 0$$

⋮

$$f_n(x_1, x_2, \dots, x_n) = 0$$

$n$  個代數方程式解  $n$  度空間的  $n$  個座標

若線性系統且不相依，則此系統只有一個平衡點

非線性系統可有多個平衡點



## Example 2

線性彈簧

$$m\ddot{x} + kx = 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1$$

平衡點在  $\bar{x}_1 = \bar{x}_2 = 0$

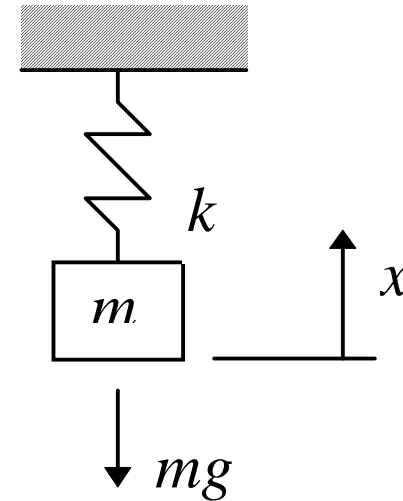
考慮重力

$$m\ddot{x} + kx = -mg$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - g$$

平衡點在  $\bar{x}_1 = -mg / k, \bar{x}_2 = 0$





## Example 3

碗裡的滾珠

$$\ddot{x}[1 + (-x + x^3)^2] + x\dot{x}^2(-1 + x^2)(-1 + 3x^2) + gx(-1 + x^2) = 0$$

此系統狀態可定義為  $x, \dot{x}$

在平衡時  $\dot{x} = \ddot{x} = 0$

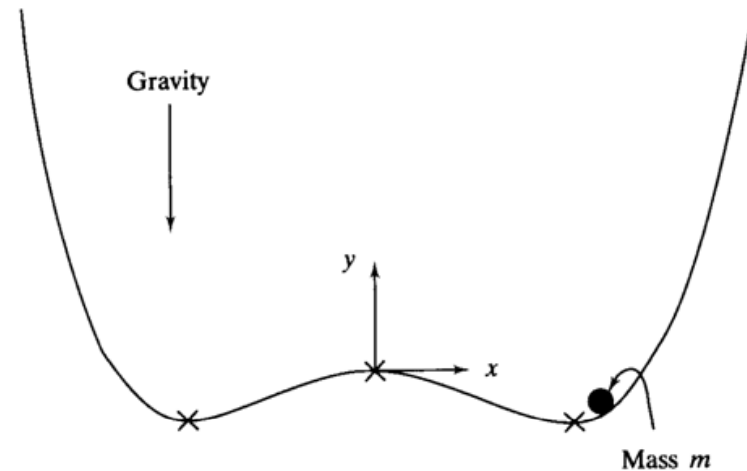
可得  $g\bar{x}(-1 + \bar{x}^2) = 0$

有三個平衡點

$$\bar{x} = 0, \bar{\dot{x}} = 0$$

$$\bar{x} = 1, \bar{\dot{x}} = 0$$

$$\bar{x} = -1, \bar{\dot{x}} = 0$$

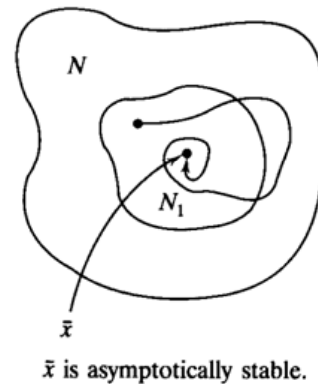
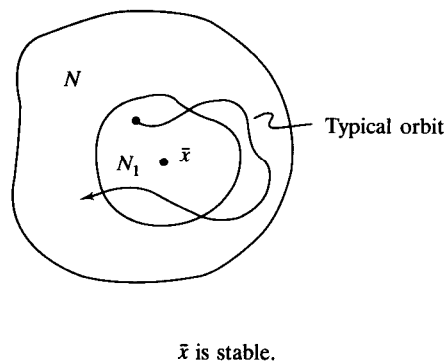




## 平衡點的穩定性

穩定性的討論分為：Local 與 Global behavior

**Stability of equilibrium:** If for every  $n$ -dimensional neighborhood  $N$ , no matter how small of  $\underline{x}$ , there is a neighborhood  $N_1 \subset N$ , such that every solution starting out in  $N_1$  lies in  $N$  for all time, then the equilibrium solution  $\underline{x}$  is stable. Furthermore, if  $N_1$  can be selected so that  $x(t) \rightarrow \underline{x}$  as  $t \rightarrow \infty$ , then  $\underline{x}$  is asymptotically stable.







# Example 4

線性彈簧

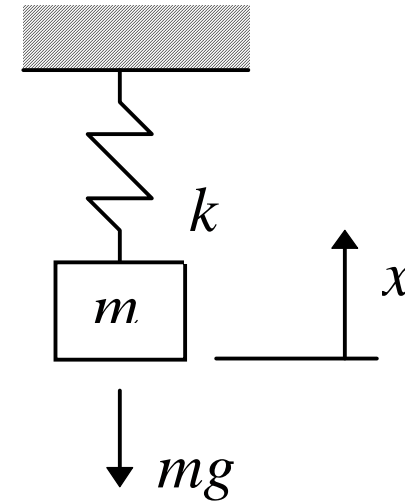
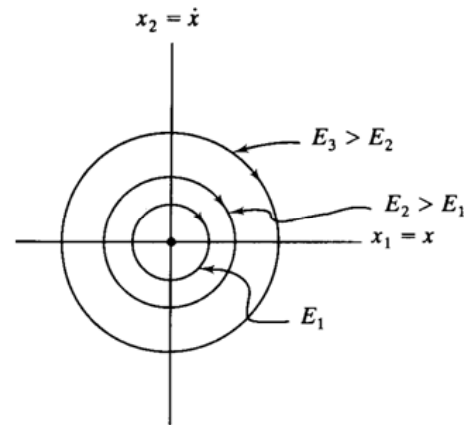
$$m\ddot{x} + kx = 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1$$

取 $m=k=1$ , 其能量為  $\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 = E$

其相圖為





## Example 5

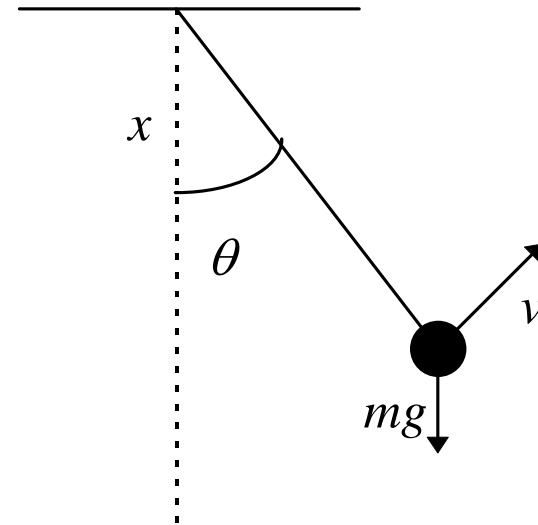
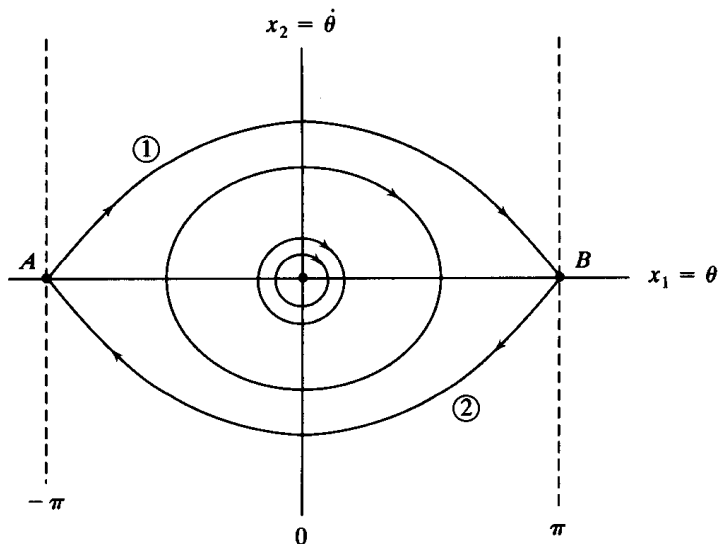
單擺

$$mL^2 \ddot{\theta} = -mgL \sin \theta$$

其能量為

$$\frac{1}{2} mL^2 \dot{\theta}^2 + mgL(1 - \cos \theta) = E = \text{constant}$$

其相圖為

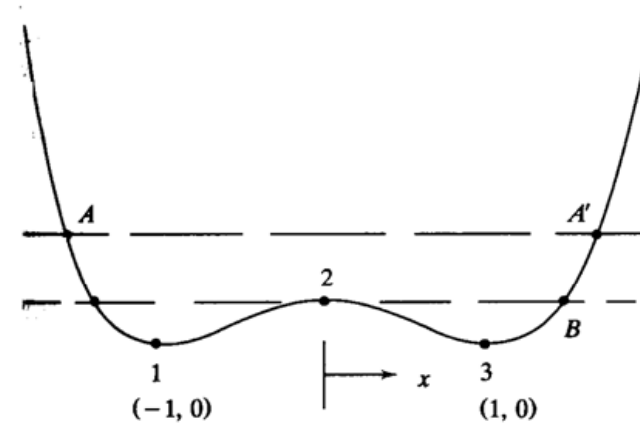




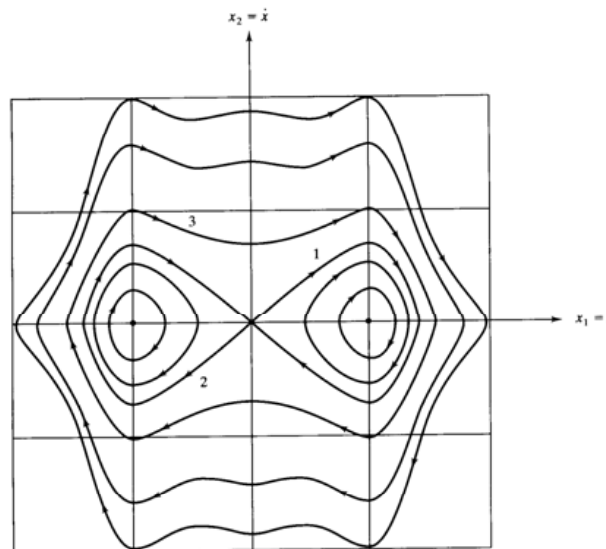
# Example 6

滾珠

三個平衡點為  
 $(-1, 0)$ ,  $(0, 0)$ ,  $(1, 0)$



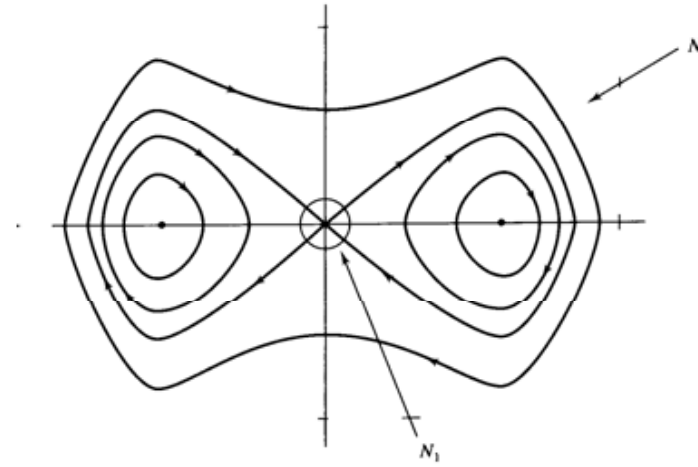
其相圖為



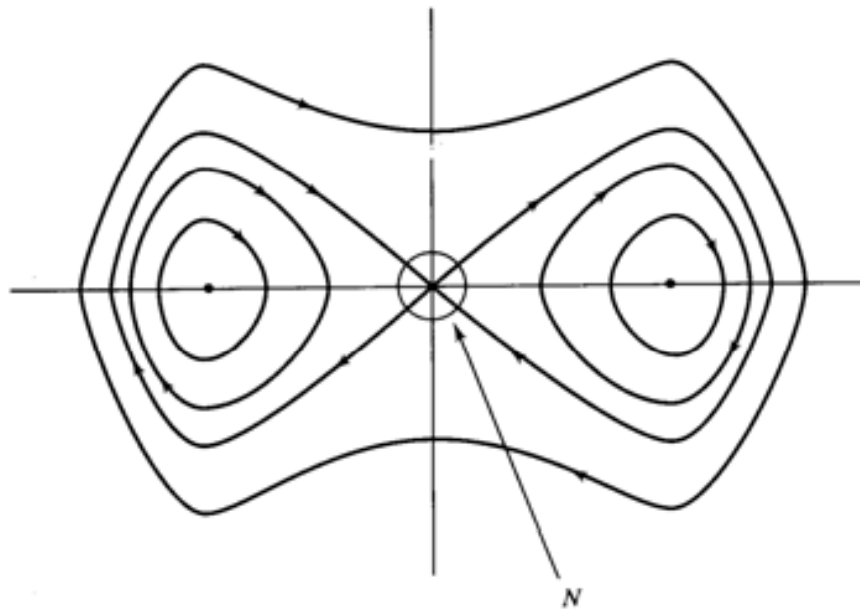


# Example 6 (cont.)

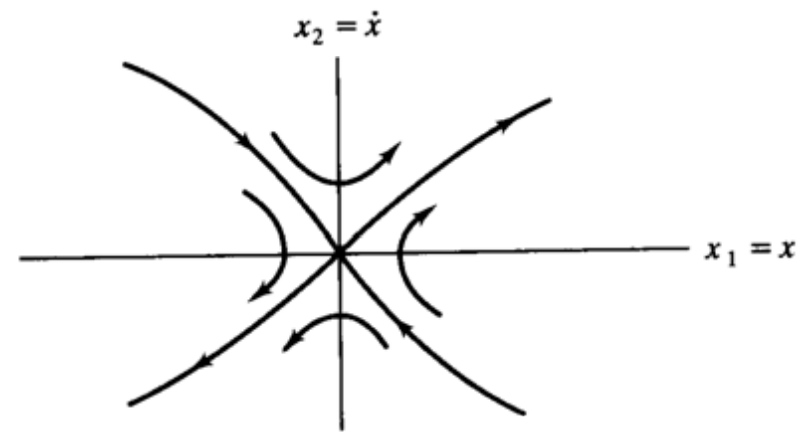
檢查其穩定性，當大的 $N$ 時



但我們要求無限小的 $N$ ，



在 $(0, 0)$ 附近，此系統不穩定。





## 系統線性化

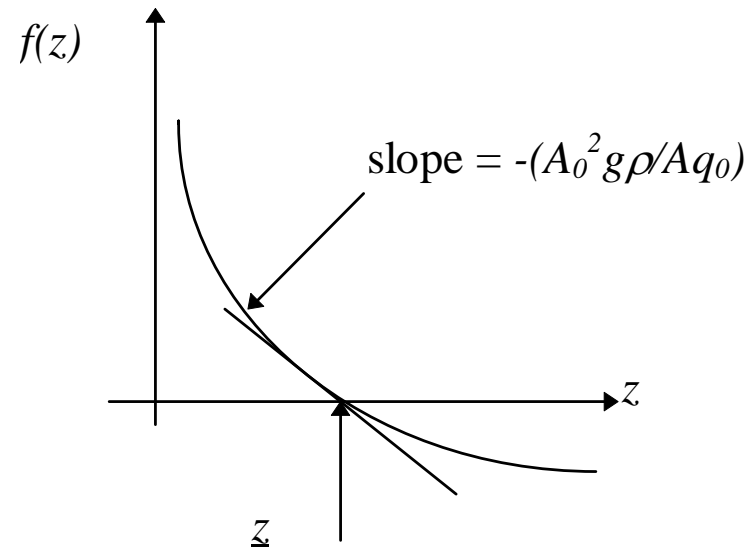
分析平衡點的穩定性基本上可分成直接分析與間接分析兩大方式，直接分析係以系統響應與一個參考函數 (Lyapunov 函數) 直接比較而判斷其穩定性。間接分析係先將系統在平衡點附近”線性化”之後再分析平衡點在此區域中的特性。線性化之後我們所考慮的是”區域性”的穩定性。

Example: 水桶

$$\dot{z} = \frac{q_0}{A\rho} - \frac{A_0}{A} (2gz)^{\frac{1}{2}}$$

其平衡點為  $\bar{z} = \frac{1}{2g} \left( \frac{q_0}{\rho A_0} \right)^2$

在某一點  $z$  時其斜率為  $\left. \frac{df}{dz} \right|_{z=\bar{z}} = -\frac{A_0 g}{A} (2g\bar{z})^{-\frac{1}{2}}$



取  $\zeta = z - \underline{z}$ ，則方程式成為  $\dot{\zeta} = -\frac{A_0^2 g \rho}{A q_0} \zeta$   
成為線性系統



## 多變數系統

系統為：

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n, t), \quad i = 1, 2, \dots, n$$

假設平衡點為  $\underline{x}$ ，則

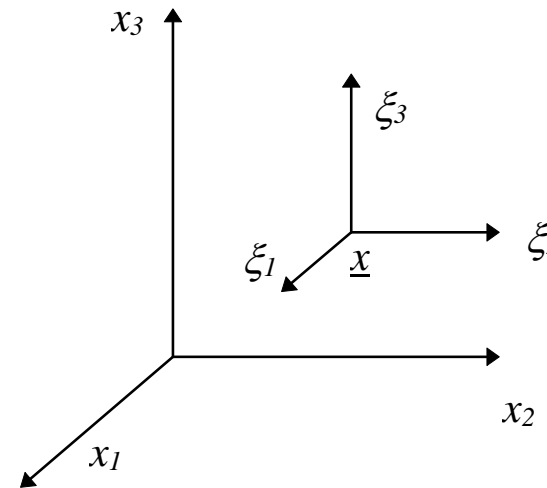
$$f_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0$$

將座標原點移到  $\underline{x}$  稱之為  $\xi$

取  $\xi_i = x_i - \bar{x}_i, \quad i = 1, \dots, n$

則

$$\dot{x}_i = \dot{\bar{x}}_i + \dot{\xi}_i = \dot{\xi}_i = f_i(\bar{x}_1 + \xi_1, \bar{x}_2 + \xi_2, \dots, \bar{x}_n + \xi_n)$$





取泰勒展開式 (Taylor series)

$$\dot{\xi}_1 = f_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) + \sum_{i=1}^n \left. \frac{\partial f_1}{\partial x_i} \right|_{\bar{x}} \xi_i + \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n \left. \frac{\partial^2 f_1}{\partial x_i \partial x_j} \right|_{\bar{x}} \xi_i \xi_j + \dots$$

$\xi_i, \xi_j$  都很小，可忽略高次項

$$\dot{\xi}_1 = a_{11}\xi_1 + a_{12}\xi_2 + \dots + a_{1n}\xi_n$$

其中  $a_{1j} = \left. \frac{\partial f_1}{\partial x_j} \right|_{\bar{x}}$

可得線性方程式

$$\begin{aligned} \dot{\xi}_1 &= a_{11}\xi_1 + a_{12}\xi_2 + \dots + a_{1n}\xi_n \\ \dot{\xi}_2 &= a_{21}\xi_1 + a_{22}\xi_2 + \dots + a_{2n}\xi_n \\ &\vdots \\ \dot{\xi}_n &= a_{n1}\xi_1 + a_{n2}\xi_2 + \dots + a_{nn}\xi_n \end{aligned}$$

或  $\dot{\xi} = \mathbf{A}\xi$





## Example

兩個狀態的線性系統

$$\dot{x}_1 = \frac{3}{2}x_1 + 10x_2$$

$$\dot{x}_2 = -40x_1 + \frac{1}{2}x_2$$

其狀態方程式可以寫成

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 10 \\ -40 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

解其”固有值” 與”固有向量”  $\begin{bmatrix} -\lambda_i + \frac{3}{2} & 10 \\ -40 & -\lambda_i + \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$

得  $\lambda_{1,2} = 1 \pm 20i$ ,  $c_1 = [1 -0.05+2i]^T$ ,  $c_2 = [1 -0.05-2i]^T$

$$\mathbf{x}(t) = \alpha_1 \begin{bmatrix} 1 \\ -0.05 + 2i \end{bmatrix} e^{(1+20i)t} + \alpha_2 \begin{bmatrix} 1 \\ -0.05 - 2i \end{bmatrix} e^{(1-20i)t}$$



## Example (cont.)

要如何考慮這種複數解呢？

事實上這類系統的解都是這種形式

$$\mathbf{x}(t) = \alpha \mathbf{c} e^{(\sigma+i\omega)t} + \alpha^* \mathbf{c}^* e^{(\sigma-i\omega)t}$$

把它的"實部" 和"虛部" 分開

$$\begin{aligned}\mathbf{x}(t) &= \frac{A}{2} e^{i\theta} [\mathbf{c}_R + i\mathbf{c}_I] e^{(\sigma+i\omega)t} + \frac{A}{2} e^{-i\theta} [\mathbf{c}_R - i\mathbf{c}_I] e^{(\sigma-i\omega)t} \\ &= \frac{A}{2} e^{\sigma t} \left\{ \mathbf{c}_R (e^{i(\omega t + \theta)} + e^{-i(\omega t + \theta)}) + i\mathbf{c}_I (e^{i(\omega t + \theta)} - e^{-i(\omega t + \theta)}) \right\} \\ &= \frac{A}{2} e^{\sigma t} \left\{ \mathbf{c}_R \cos(\omega t + \theta) - \mathbf{c}_I \sin(\omega t + \theta) \right\}\end{aligned}$$



## Example (cont.)

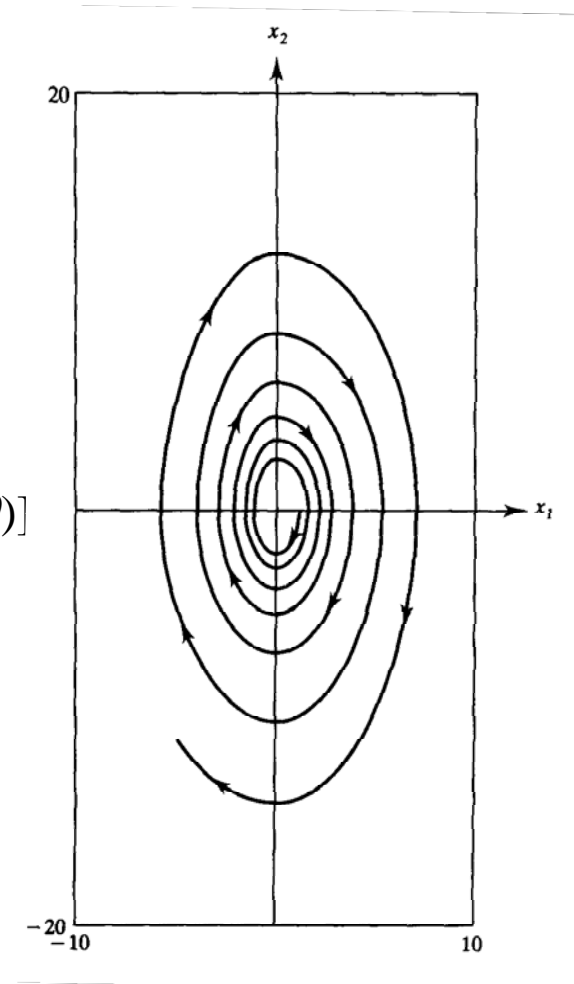
上例中

$$\mathbf{c}_R = \begin{bmatrix} 1 \\ -0.05 \end{bmatrix}, \quad \mathbf{c}_I = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \sigma = 1, \quad \omega = 20$$

得

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} Ae^t \cos(20t + \theta) \\ Ae^t [-0.05 \cos(20t + \theta) - 2 \sin(20t + \theta)] \end{bmatrix}$$

A 和  $\theta$  由起始條件決定，其相圖為





## 二階系統

狀態方程式：

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

其特徵方程式為

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

或寫成  $\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$

或  $\lambda^2 - T\lambda + \Delta = 0$

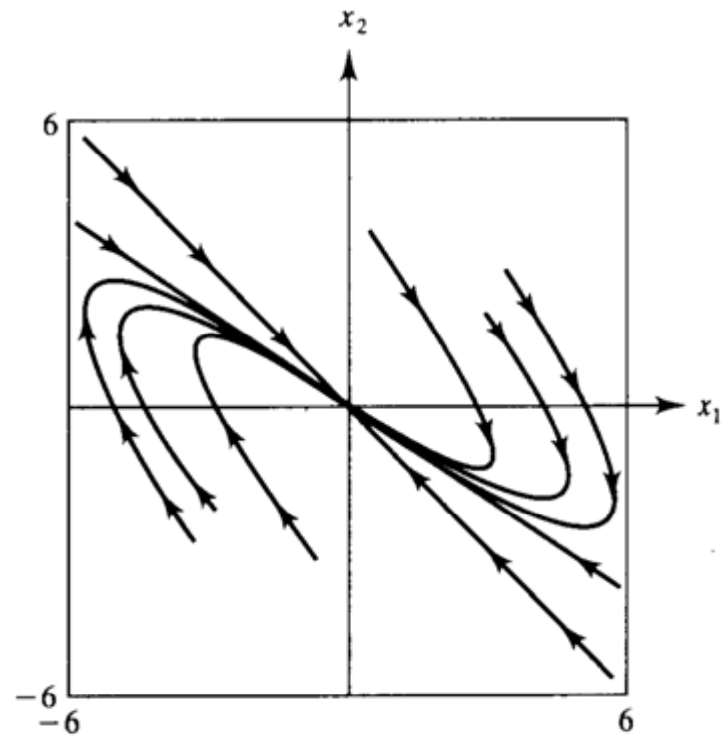
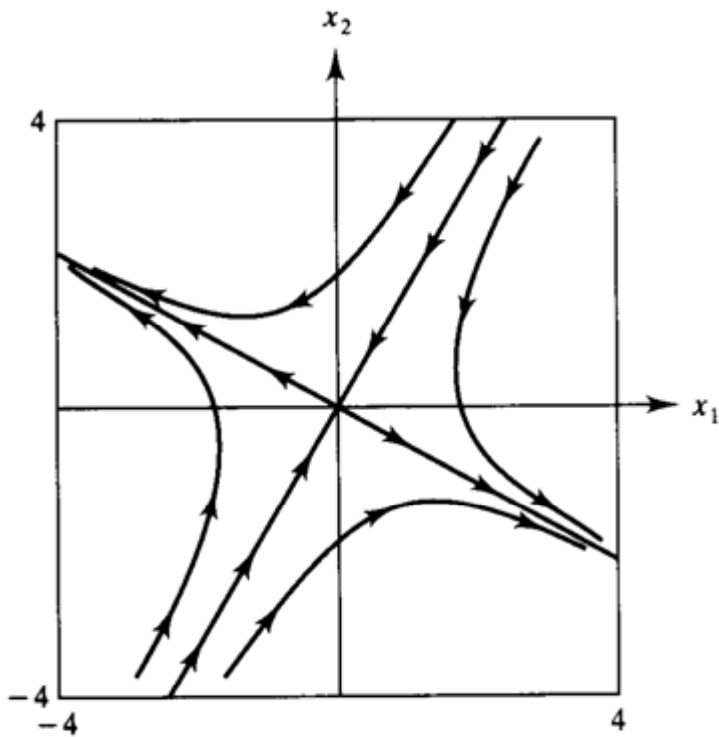
其固有值解為  $\lambda_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - \Delta}$



這些變化與固有值間的關係如下：

$$\mathbf{A} = \begin{bmatrix} 0.4 & -1.2 \\ -1.2 & -1.4 \end{bmatrix}, \quad \lambda_1 = -2, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_2 = 1, \mathbf{c}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}, \quad \lambda_1 = -2, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_2 = -1, \mathbf{c}_2 = \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

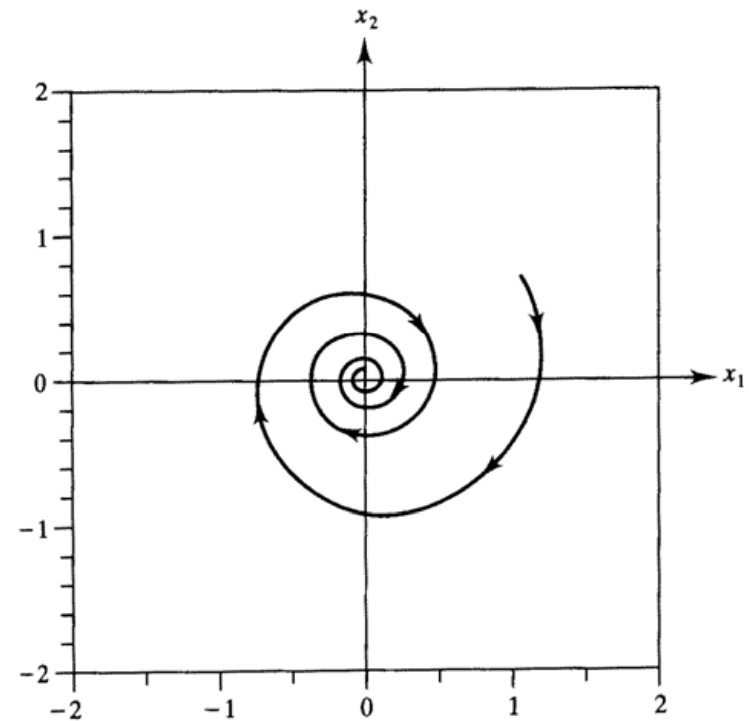
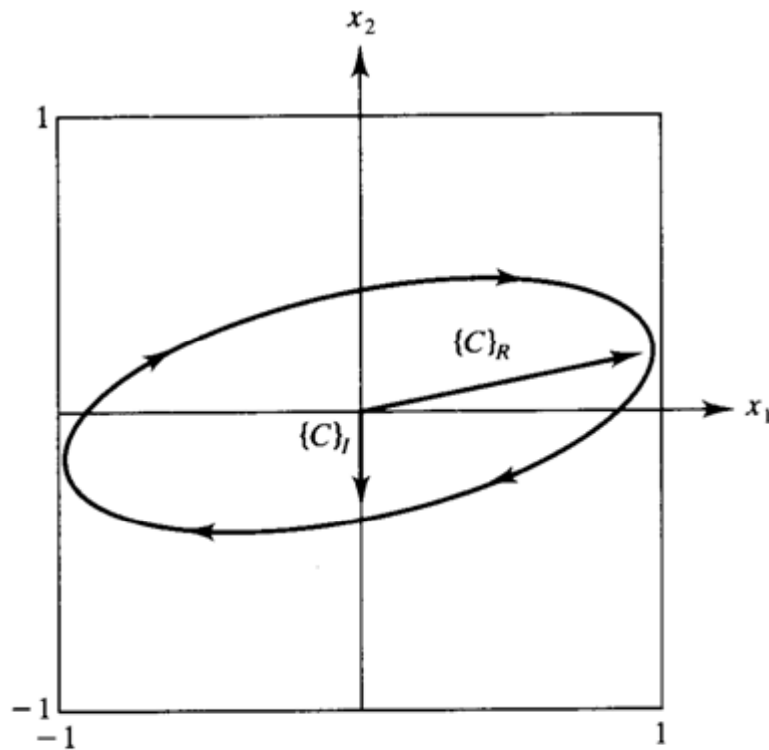




這些變化與固有值間的關係如下：

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix}, \quad \lambda_{1,2} = \pm 2i, \quad \mathbf{c}_R = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \quad \mathbf{c}_I = \begin{bmatrix} 1 \\ -0.4 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 7 \\ -7 & -1 \end{bmatrix}, \quad \lambda_{1,2} = -1 \pm 7i, \quad \mathbf{c}_R = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_I = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$





$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad \lambda_1 = 0, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

