## § core region --- perturbation method

(1) characteristic velocity

$$u_c^* = \frac{g\beta\Delta T H^3}{\nu L} = \frac{\alpha}{L} Ra_H \quad \text{and} \quad v_c^* = u_c^* \frac{H}{L} = \varepsilon^{1/2} \cdot \frac{\alpha}{L} Ra_H \text{, with } \varepsilon \equiv \left(\frac{H}{L}\right)^2$$

(2) Nondimensionalize

$$u \mapsto u/u_c^* = \frac{uL}{\alpha} Ra_H^{-1} \quad \text{and} \quad v \mapsto v/v_c^* = \varepsilon^{-1/2} \cdot \frac{v\alpha}{L} Ra_H^{-1}$$

$$x \mapsto x/L \quad \text{and} \quad y \mapsto y/H$$

$$T \mapsto \frac{T - T_{cold}}{\Delta T} \quad \text{such that } T(x=0)=0 \quad \text{and} \quad T(x=L)=1$$

(3) Dimensional equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \nu\left\{\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right\} + g\beta\frac{\partial T}{\partial x}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right\}$$

(4) Dimensionless equations:

(a) $$\frac{u_c^*}{L}\frac{\partial u}{\partial x} + \frac{v_c^*}{H}\frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \underline{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

(b) $$\underbrace{\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)}_{\frac{u_c^* v_c^*}{L^2}} - \underbrace{\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)}_{\frac{u_c^{*2}}{LH}} = \nu\left\{\frac{\partial}{\partial x}\left(\underset{\frac{\nu v_c^*}{L^3}}{\frac{\partial^2 v}{\partial x^2}} + \underset{\frac{\nu v_c^*}{LH^2}}{\frac{\partial^2 v}{\partial y^2}}\right) - \frac{\partial}{\partial y}\left(\underset{\frac{\nu u_c^*}{HL^2}}{\frac{\partial^2 u}{\partial x^2}} + \underset{\frac{\nu u_c^*}{H^3}}{\frac{\partial^2 u}{\partial y^2}}\right)\right\} + \underbrace{g\beta\frac{\partial T}{\partial x}}_{\frac{g\beta\Delta T}{L}}$$

- $$\frac{u_c^* v_c^* / L^2}{u_c^{*2} / LH} = \frac{v_c^*}{u_c^*} \cdot \frac{H}{L} = \varepsilon$$
- $$\frac{\nu v_c^* / L^3}{u_c^{*2} / LH} = \frac{\varepsilon^{1/2}}{u_c^*} \cdot \frac{\nu H}{L^2} = \varepsilon^{1/2} \frac{L}{\alpha} Ra_H^{-1} \cdot \frac{\nu H}{L^2} = \varepsilon \Pr Ra_H^{-1}$$

- $$\frac{\nu v_c^* / LH^2}{u_c^{*2} / LH} = \frac{\varepsilon^{1/2}}{u_c^*} \cdot \frac{\nu}{H} = \varepsilon^{1/2} \frac{L}{\alpha} Ra_H^{-1} \cdot \frac{\nu}{H} = \Pr Ra_H^{-1}$$

- $$\frac{\nu u_c^* / HL^2}{u_c^{*2} / LH} = \frac{1}{u_c^*} \cdot \frac{\nu}{L} = \frac{L}{\alpha} Ra_H^{-1} \cdot \frac{\nu}{L} = \Pr Ra_H^{-1}$$

- $$\frac{\nu u_c^* / H^3}{u_c^{*2} / LH} = \frac{1}{u_c^*} \cdot \frac{\nu L}{H^2} = \frac{L}{\alpha} Ra_H^{-1} \cdot \frac{\nu L}{H^2} = \varepsilon^{-1} \Pr Ra_H^{-1}$$

- $$\frac{g\beta\Delta T / L}{u_c^{*2} / LH} = \left( \frac{\nu L}{g\beta\Delta T H^3} \right)^2 \cdot \frac{g\beta\Delta T H}{1} = \frac{\nu^2 L^2}{g\beta\Delta T H^5} = \frac{\alpha\nu}{g\beta\Delta T H^3} \cdot \frac{\nu}{\alpha} \cdot \frac{L^2}{H^2} = \varepsilon^{-1} \Pr Ra_H^{-1}$$

Therefore

$$\frac{Ra_H}{\Pr} \left\{ \varepsilon \frac{\partial}{\partial x} \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \right\} = \left\{ \frac{\partial}{\partial x} \left( \varepsilon \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial}{\partial y} \left( \frac{\partial^2 u}{\partial x^2} + \varepsilon^{-1} \frac{\partial^2 u}{\partial y^2} \right) \right\} + \varepsilon^{-1} \frac{\partial T}{\partial x}$$

or

$$\boxed{\varepsilon \frac{Ra_H}{\Pr} \left\{ \varepsilon \frac{\partial}{\partial x} \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \right\} = \varepsilon \frac{\partial}{\partial x} \left( \varepsilon \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial}{\partial y} \left( \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial T}{\partial x}}$$

(c) $$\underbrace{u \frac{\partial T}{\partial x}}_{\frac{u_c^* \Delta T}{L}} + \underbrace{v \frac{\partial T}{\partial y}}_{\frac{v_c^* \Delta T}{H}} = \alpha \left\{ \underbrace{\frac{\partial^2 T}{\partial x^2}}_{\frac{\alpha \Delta T}{L^2}} + \underbrace{\frac{\partial^2 T}{\partial y^2}}_{\frac{\alpha \Delta T}{H^2}} \right\}$$

- $$\frac{u_c^* \Delta T / L}{\alpha \Delta T / H^2} = \frac{\alpha}{L} Ra_H \frac{H^2}{\alpha L} = \varepsilon Ra_H$$

- $$\frac{v_c^* \Delta T / H}{\alpha \Delta T / H^2} = \varepsilon^{1/2} \frac{\alpha}{L} Ra_H \frac{H}{\alpha} = \varepsilon Ra_H$$

- $\dfrac{\alpha\Delta T / L^2}{\alpha\Delta T / H^2} = \dfrac{H^2}{L^2} = \varepsilon$

Therefore

$$\varepsilon Ra_H \left\{ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right\} = \varepsilon\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(5) perturbation solution

$$(u, v, T) = (u_0, v_0, T_0) + \varepsilon(u_1, v_1, T_1) + \varepsilon^2(u_2, v_2, T_2) + \cdots$$

Substitute into dimensionless equations:

(a) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \;\Rightarrow\; \left(\dfrac{\partial u_0}{\partial x} + \dfrac{\partial v_0}{\partial y}\right) + \varepsilon\left(\dfrac{\partial u_1}{\partial x} + \dfrac{\partial v_1}{\partial y}\right) + \varepsilon^2\left(\dfrac{\partial u_2}{\partial x} + \dfrac{\partial v_2}{\partial y}\right) + \cdots$

(b)

$$\varepsilon\frac{Ra_H}{\Pr}\left\{ \varepsilon\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right\} = \varepsilon\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial T}{\partial x}$$

$$\Downarrow$$

$$-\varepsilon\frac{Ra_H}{\Pr}\frac{\partial}{\partial y}\left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}\right) + \varepsilon^2\frac{Ra_H}{\Pr}\left\{\frac{\partial}{\partial x}\left(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y}\right) - \frac{\partial}{\partial y}\left(u_1\frac{\partial u_0}{\partial x} + v_1\frac{\partial u_0}{\partial y} + u_0\frac{\partial u_1}{\partial x} + v_0\frac{\partial u_1}{\partial y}\right)\right\}$$

$$+\cdots = -\frac{\partial^3 u_0}{\partial y^3} + \frac{\partial T_0}{\partial x} + \varepsilon\left\{\frac{\partial^3 v_0}{\partial x\partial y^2} - \frac{\partial^3 u_0}{\partial y\partial x^2} - \frac{\partial^3 u_1}{\partial y^3} + \frac{\partial T_1}{\partial x}\right\} + \varepsilon^2\left\{\frac{\partial^3 v_0}{\partial x^3} + \frac{\partial^3 v_1}{\partial x\partial y^2} - \frac{\partial^3 u_1}{\partial y\partial x^2} - \frac{\partial^3 u_2}{\partial y^3} + \frac{\partial T_2}{\partial x}\right\} + \cdots$$

(c)

$$\varepsilon Ra_H \left\{ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right\} = \varepsilon\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$\Downarrow$$

$$\varepsilon Ra_H\left\{u_0\frac{\partial T_0}{\partial x} + v_0\frac{\partial T_0}{\partial y}\right\} + \varepsilon^2 Ra_H\left\{u_0\frac{\partial T_1}{\partial x} + v_0\frac{\partial T_1}{\partial y} + u_1\frac{\partial T_0}{\partial x} + v_1\frac{\partial T_0}{\partial y}\right\} + \cdots$$

$$= \frac{\partial^2 T_0}{\partial y^2} + \varepsilon\left(\frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2}\right) + \varepsilon^2\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2}\right) + \cdots$$

(6) Terms multiplied by the same power of ε are grouped together and set equal to zero:

(a) $\varepsilon^0$ :

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \qquad (1)$$

$$-\frac{\partial^3 u_0}{\partial y^3} + \frac{\partial T_0}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial^2 T_0}{\partial y^2} = 0 \qquad (3)$$

(b) $\varepsilon^1$ :

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \qquad (1)$$

$$-\frac{Ra_H}{\Pr}\frac{\partial}{\partial y}\left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}\right) = \frac{\partial^3 v_0}{\partial x \partial y^2} - \frac{\partial^3 u_0}{\partial y \partial x^2} - \frac{\partial^3 u_1}{\partial y^3} + \frac{\partial T_1}{\partial x} \qquad (2)$$

$$Ra_H\left\{u_0\frac{\partial T_0}{\partial x} + v_0\frac{\partial T_0}{\partial y}\right\} = \frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} \qquad (3)$$

(7) proper boundary conditions

$$u_i = v_i = 0 \quad \text{at} \quad y = 0 \text{ and } y = 1$$

$$\frac{\partial T_i}{\partial y} = 0 \quad \text{at} \quad y = 0 \text{ and } y = 1$$

$$\int_0^1 u_i dy = 0 \quad \text{for mass conservation}$$

for all $i$

(8) Solve solutions systematically

(a)

| | |
|---|---|
| $\frac{\partial^2 T_0}{\partial y^2}=0 \;\Rightarrow\; T_0(x,y)=a(x)y+b(x)$<br><br>$BC \Rightarrow a(x)=0 \Rightarrow T_0(x,y)=b(x)=T_0(x)$ | $\frac{\partial^3 u_0}{\partial y^3}=\frac{\partial T_0}{\partial x}=b'$<br>$\Rightarrow u_0(x,y)=\frac{b'}{6}y^3+c(x)y^2+d(x)y+e(x)$<br>BC's $\Rightarrow e(x)=0$<br>$\frac{b'}{6}+c+d=0$<br>$\frac{b'}{24}+\frac{c}{3}+\frac{d}{2}=0$<br>$\Rightarrow u_0(x,y)=b'(x)\left(\frac{y^3}{6}-\frac{y^2}{4}+\frac{y}{12}\right)$ |
| $-\frac{\partial v_0}{\partial y}=\frac{\partial u_0}{\partial x}=b''(x)\left(\frac{y^3}{6}-\frac{y^2}{4}+\frac{y}{12}\right)$<br>$\Rightarrow -v_0(x,y)=b''(x)\left(\frac{y^4}{24}-\frac{y^3}{12}+\frac{y^2}{24}\right)+f(x)$<br>$BC's \Rightarrow f(x)=0$ and $b''(x)=0$<br><br>$\Rightarrow v_0(x,y)=0$<br><br>$b(x)=K_1x+K_2$ | In remark:<br>$T_0(x,y)=K_1x+K_2$<br>$u_0(x,y)=K_1\left(\frac{y^3}{6}-\frac{y^2}{4}+\frac{y}{12}\right)$<br>$v_0(x,y)=0$ |

(c) Notice that $v_0(x,y)=0$, $u_0(x,y)=u_0(y)$ and $T_0(x,y)=T_0(x)$ linear in $x$. The equations are reduced to

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0 \qquad (1)$$

$$0=-\frac{\partial^3 u_1}{\partial y^3}+\frac{\partial T_1}{\partial x} \qquad (2)$$

$$Ra_H u_0\frac{\partial T_0}{\partial x}=\frac{\partial^2 T_1}{\partial y^2} \qquad (3)$$

| | |
|---|---|
| $\frac{\partial^2 T_1}{\partial y^2}=Ra_H u_0\frac{\partial T_0}{\partial x}=K_1^2 Ra_H\left(\frac{y^3}{6}-\frac{y^2}{4}+\frac{y}{12}\right)$<br><br>$\Rightarrow T_1(x,y)=K_1^2 Ra_H\left(\frac{y^5}{120}-\frac{y^4}{48}+\frac{y^3}{72}\right)+a(x)y+b(x)$<br>$BC's \Rightarrow a(x)=0$ | $\frac{\partial^3 u_1}{\partial y^3}=\frac{\partial T_1}{\partial x}=b'(x)$<br>$\Rightarrow u_1(x,y)=b'(x)\left(\frac{y^3}{6}-\frac{y^2}{4}+\frac{y}{12}\right)$<br>same as (a) |

In remark:

$$T_1(x,y) = K_1^2 Ra_H \left( \frac{y^5}{120} - \frac{y^4}{48} + \frac{y^3}{72} \right) + K_3 x + K_4$$

$$u_1(x,y) = K_3 \left( \frac{y^3}{6} - \frac{y^2}{4} + \frac{y}{12} \right)$$

$$v_1(x,y) = 0$$

It turns out **the systematic solution has the same form at all orders in ε.** Therefore,

$$T(x,y) = K_1 x + K_2 + \left( \frac{H}{L} \right)^2 K_1^2 Ra_H \left( \frac{y^5}{120} - \frac{y^4}{48} + \frac{y^3}{72} \right)$$

$$u(x,y) = K_1 \left( \frac{y^3}{6} - \frac{y^2}{4} + \frac{y}{12} \right)$$

$$v(x,y) = 0$$

$$K_1 = C_1 + C_2 \left( \frac{H}{L} \right) + C_3 \left( \frac{H}{L} \right)^2 + \cdots$$

$$K_2 = C_1^* + C_2^* \left( \frac{H}{L} \right) + C_3^* \left( \frac{H}{L} \right)^2 + \cdots$$

(new definitions of $K_1$ and $K_2$)

Or in dimensional form:

$$\frac{T - T_{cold}}{\Delta T} = K_1 \frac{x}{L} + K_2 + K_1^2 \left( \frac{H^2}{L^2} Ra_H \right) \left\{ \frac{1}{120} \left( \frac{y}{H} \right)^5 - \frac{1}{48} \left( \frac{y}{H} \right)^4 + \frac{1}{72} \left( \frac{y}{H} \right)^3 \right\}$$

$$\frac{u}{\alpha Ra_H / L} = K_1 \left( \frac{1}{6} \left( \frac{y}{H} \right)^3 - \frac{1}{4} \left( \frac{y}{H} \right)^2 + \frac{1}{12} \left( \frac{y}{H} \right)^3 \right)$$

$$\frac{v}{\alpha H Ra_H / L^2} = 0$$

§**end regions**

matching conditions: $v\ ,\ \dfrac{\partial v}{\partial x}\to 0\quad \text{as}\quad x\to\delta H$

boundary conditions: $u,v=0\ \text{ at }\ x=0\ \text{ and }\ y=0,\ y=H$

(1) integral analysis

(a) momentum equation

$$\int_0^{\delta H}\int_0^{H}\left\{\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\nu\left\{\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right\}+g\beta\frac{\partial T}{\partial x}\right\}dydx$$

Noticed that

$$u\frac{\partial v}{\partial x}\bigg|_0^{\delta H}=0\quad\because\ u=0\ \text{ at }\ x=0\ \text{(BC) and }\frac{\partial v}{\partial x}=0\ \text{ at }\ x=\delta H\ \text{(MC)}$$

$$v\frac{\partial v}{\partial y}\bigg|_0^{\delta H}=0\quad\because\ v=0\ \text{ at }\ x=0\ \text{(BC) and }\frac{\partial v}{\partial y}=0\ \text{ at }\ x=\delta H\ \text{(MC)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\bigg|_0^{H}=0\quad\because\ u,v=0\ \text{ at }\ y=0\ \text{ and }y=H\ \text{(BCs)}$$

Moreover, because of

$$\frac{\partial v}{\partial y},\frac{\partial^2 v}{\partial y^2}=0\ \text{ at }\ x=0\ \text{(BC) and }\ x=\delta H\ \text{(MC)}$$

$$\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}=0\ \text{ at }y=0\ \text{ and }y=H\ \text{(BCs)}$$

Therefore

$$0=\int_0^{H}\frac{\partial^2 v}{\partial x^2}\bigg|_0^{\delta H}dy-\nu\int_0^{\delta H}\frac{\partial^2 u}{\partial y^2}\bigg|_0^{H}dx+g\beta\int_0^{H}\left(T(\delta H,y)-T_{cold}\right)dy$$

Or in dimensionless form

$$\int_0^{1}T_e(\delta,y_c)dy_c=-\int_0^{1}\frac{\partial^2 v_e}{\partial x_1^2}\bigg|_0^{\delta}dy_c+\int_0^{\delta}\frac{\partial^2 u_e}{\partial y_c^2}\bigg|_0^{1}dx$$

where

$$u_e, v_e \equiv u/u_e^* \ , \ v/u_e^*$$
$$u_e^* \equiv g\beta H^2 \Delta T/\nu = \frac{\alpha}{H} Ra_H$$
$$T_e \equiv (T - T_{cold})/\Delta T$$
$$x_1 \equiv x/H \ , \quad y_c \equiv y/H$$

(b) energy equation: $\int_0^{\delta H}\int_0^H \left\{ u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right\}\right\} dydx$

Noticed that

$vT = 0$ at $y = 0$ and $y = H$ (BCs)

$\frac{\partial T}{\partial x} \to \frac{K_1 \Delta T}{L}$ as $x \to \delta H$ (MC)

$\frac{\partial T}{\partial y} = 0$ at $y = 0$ , $y = H$ (BCs)

Therefore

$$\int_0^H uT(\delta H)dy = \alpha \int_0^H \left\{ \frac{K_1 \Delta T}{L} - \frac{\partial T}{\partial x}(0, y) \right\} dy$$

Or in dimensionless form:

$$Ra_H \int_0^1 u_e T_e(\delta, y_c) dy_c = K_1 \frac{H}{L} - \int_0^1 \frac{\partial T_e}{\partial x_1}(0, y_c) dy_c$$

(2) Guess reasonable profiles for velocity and temperature in the end regions:

(a) x-velocity:

$$u_e = 0 \quad \text{at} \quad y_c = 0 \, , \, y_c = 1$$
$$u_e \to \frac{H}{L} u_c = K_1 \frac{H}{L} \left( \frac{y_c^{\ 3}}{6} - \frac{y_c^{\ 2}}{4} + \frac{y_c}{12} \right) \quad \text{as } x_1 \to \delta$$

guess $u_e = K_1 \frac{H}{L} f\left(\frac{x_1}{\delta}\right)\left(\frac{y_c^{\ 3}}{6} - \frac{y_c^{\ 2}}{4} + \frac{y_c}{12}\right)$ with the following constraints on *f*:

$$f(0)=0 \quad \Leftrightarrow \quad u_e=0 \text{ at } x_1=0$$

$$f(1)=1 \quad \Leftrightarrow \quad u_e \to \frac{H}{L}u_c \text{ as } x_1 \to \delta$$

$$f'(0)=0 \quad \Leftrightarrow \quad v_e(0,y_c)=0 \text{ at } x_1=0+\text{continuity eqn}$$

$$f'(1)=0 \quad \Leftrightarrow \quad \frac{\partial u_e}{\partial x_1}=0 \quad \text{as } x_1 \to \delta$$

$$f''(1)=0 \quad \Leftrightarrow \quad \frac{\partial^2 u_e}{\partial x_1^2}=0 \quad \text{as } x_1 \to \delta$$

Thus

$$u_e = K_1 \frac{H}{L}\left(\frac{x_1}{\delta}\right)^2\left(6-8\frac{x_1}{\delta}+3\left(\frac{x_1}{\delta}\right)^2\right)\left(\frac{y_c{}^3}{6}-\frac{y_c{}^2}{4}+\frac{y_c}{12}\right)$$

(b) y-velocity: from continuity equation

$$\frac{\partial u_e}{\partial x_1}+\frac{\partial v_e}{\partial y_c}=0 \;\Rightarrow\; \frac{\partial v_e}{\partial y_c}=-\frac{\partial u_e}{\partial x_1}=12\frac{K_1}{\delta}\frac{H}{L}\frac{x_1}{\delta}\left(1-\frac{x_1}{\delta}\right)^2\left(\frac{y_c{}^3}{6}-\frac{y_c{}^2}{4}+\frac{y_c}{12}\right)$$

$$\Rightarrow v_e=-\frac{K_1}{\delta}\frac{H}{L}\frac{x_1}{\delta}\left(1-\frac{x_1}{\delta}\right)^2\left(\frac{y_c{}^4}{2}-y_c{}^3+\frac{y_c^2}{2}\right)+g(x_1)$$

boundary conditions: $v_e=0$ at $x=0$ and $y=0$ , $y=H$ $\Rightarrow$ $g(x_1)=0$

$$v_e=-\frac{K_1}{\delta}\frac{H}{L}\frac{x_1}{\delta}\left(1-\frac{x_1}{\delta}\right)^2\left(\frac{y_c{}^4}{2}-y_c{}^3+\frac{y_c^2}{2}\right)$$

**(c)** temperature profile:

$$T_e=0 \quad \text{at} \quad x_1=0$$

$$\frac{\partial T_e}{\partial y_c}=0 \text{ at } y_c=0 \text{ , } y_c=1$$

$$T_e \Leftrightarrow T_c=K_1x_c+K_2+\left(\frac{K_1H}{L}\right)^2 Ra_H\left(\frac{y^5}{120}-\frac{y^4}{48}+\frac{y^3}{72}\right) \text{ as } x_1 \to \delta$$

guess
$$T_e=\underbrace{\left\{T_c(\delta,y_c)-\delta\frac{K_1H}{L}\right\}h\left(\frac{x_1}{\delta}\right)}_{\text{correction part}}+\underbrace{\frac{K_1H}{L}x_1}_{\text{linear part}} \qquad \text{with}$$

$$h(0)=0 \quad \Leftrightarrow \quad T_e=0 \text{ at } x_1=0$$

$$h(1)=1 \quad \Leftrightarrow \quad T_e \to T_c(\delta,y) \text{ as } x_1 \to \delta$$

$$h'(1)=0 \quad \Leftrightarrow \quad \frac{\partial T_e}{\partial x_1}\to\frac{\partial T_c}{\partial x_1}=\frac{K_1H}{L} \text{ as } x_1 \to \delta$$

Consequently,

$$T_e = \underbrace{\left\{ T_c\left(\delta, y_c\right) - \delta \frac{K_1 H}{L} \right\} \left( 2\frac{x_1}{\delta} - \left(\frac{x_1}{\delta}\right)^2 \right)}_{\text{correction part}} + \underbrace{\frac{K_1 H}{L} x_1}_{\text{linear part}}$$

(3) results: substitute (2) into (1):

$$K_2 + \frac{\delta H K_1}{L} + \frac{H^2 K_1^2 Ra_H}{1440 L^2} = \frac{H K_1}{10\delta^3 L} + \frac{3\delta H K_1}{5L}$$

$$-\left(\frac{HK_1}{L}\right)^3 \frac{Ra_H^2}{362880} = \frac{HK_1}{L} - \left\{ \frac{2K_2}{\delta} + \frac{HK_1}{L} + \left(\frac{HK_1}{L}\right)^2 \frac{Ra_H}{720\delta} \right\}$$

after rearrangement:

$$K_2 + \left(\frac{HK_1}{L}\right)^2 \frac{Ra_H}{1440} = \frac{2\delta H K_1}{5L}\left(\frac{1}{4\delta^4} - 1\right)$$

$$\left(\frac{HK_1}{L}\right)^3 \frac{\delta Ra_H^2}{725760} = K_2 + \left(\frac{HK_1}{L}\right)^2 \frac{Ra_H}{1440}$$

two equations for three unknowns: $K_1$, $K_2$ , $\delta$ for given values of $\frac{H}{L}$ and $Ra_H$ .

(4) third equation: Noticed that the flow is symmetric about the geometric center of the cavity. Thus

$$T_c = \frac{1}{2} \quad \text{at} \quad x_c = \frac{1}{2} \text{ and } y_c = \frac{1}{2}$$

i.e. $$\frac{K_1}{2} + K_2 + \left(K_1 \frac{H}{L}\right)^2 \frac{Ra_H}{1440} = \frac{1}{2}$$

(5) solve the system of algebraic equations: define

$$\Omega \equiv K_2 + \left(\frac{HK_1}{L}\right)^2 \frac{Ra_H}{1440}$$

and rewrite equations as follows

$$\Omega = \frac{2\delta H K_1}{5L}\left(\frac{1}{4\delta^4}-1\right) = \delta\left\{\left(\frac{HK_1}{L}\right)^3 \frac{Ra_H^2}{725760}\right\} = \frac{1}{2}-\frac{K_1}{2}$$

thus

(a) $\delta = \left\{\left(\frac{HK_1}{L}\right)^3 \frac{Ra_H^2}{725760}\right\}^{-1}\left(\frac{1}{2}-\frac{K_1}{2}\right)$

(b) $\frac{2\delta H K_1}{5L}\left(\frac{1}{4\delta^4}-1\right) = \frac{1}{2}-\frac{K_1}{2}$

(c) $\Omega = K_2 + \left(\frac{HK_1}{L}\right)^2 \frac{Ra_H}{1440} = \frac{1}{2}-\frac{K_1}{2}$

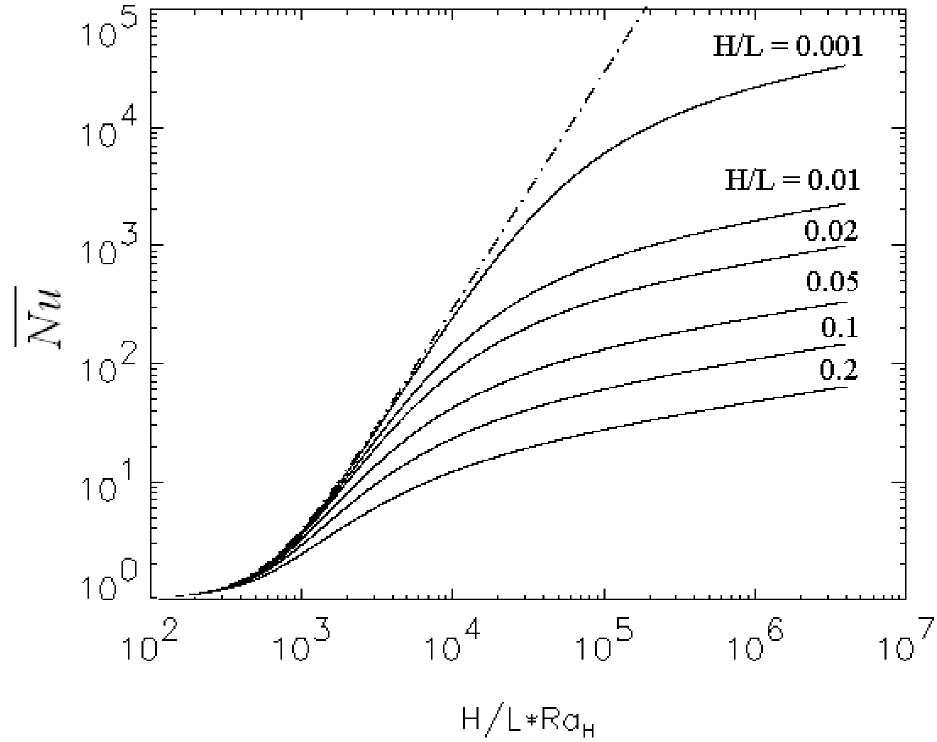
⇨ given $\frac{H}{L}$ and $Ra_H$ ⇨ find $K_1$ by root-searching (b) with δ replaced by (a)

(6) The conduction-referenced Nusselt number:

$$q' \equiv k\int_{-H/2}^{H/2}\left(-\frac{\partial T}{\partial x}\right)_{x=0} dy$$

$$\overline{Nu} \equiv \frac{q'}{(kH\Delta T)/L} = K_1 + \frac{K_1^3}{362880}\left(\frac{H}{L}Ra_H^2\right)^2$$

(in terms of dimensional variables)



1. $$T_c(\delta, y_c) = \frac{K_1 \delta H}{L} + K_2 + \left(\frac{K_1 H}{L}\right)^2 Ra_H \left(\frac{y_c^5}{120} - \frac{y_c^4}{48} + \frac{y_c^3}{72}\right)$$

2. $$T_e = \left\{T_c(\delta, y_c) - \delta \frac{K_1 H}{L}\right\}\left(2\frac{x_1}{\delta} - \left(\frac{x_1}{\delta}\right)^2\right) + \frac{K_1 H}{L} x_1$$

3. $$\left(\frac{\partial T_e}{\partial x_1}\right)_{x_1=0} = \left\{T_c(\delta, y_c) - \delta \frac{K_1 H}{L}\right\}\frac{2}{\delta} + \frac{K_1 H}{L} = \frac{2T_c(\delta, y_c)}{\delta} - \frac{K_1 H}{L}$$

$$= \frac{K_1 H}{L} + \frac{2K_2}{\delta} + \frac{2}{\delta}\left(\frac{K_1 H}{L}\right)^2 Ra_H \left(\frac{y_c^5}{120} - \frac{y_c^4}{48} + \frac{y_c^3}{72}\right)$$

4. $$\int_{-1/2}^{1/2}\left(\frac{\partial T_e}{\partial x_1}\right)_{x_1=0} dy_c = \int_{-1/2}^{1/2}\left\{\frac{K_1 H}{L} + \frac{2K_2}{\delta} + \frac{2}{\delta}\left(\frac{K_1 H}{L}\right)^2 Ra_H \left(\frac{y_c^5}{120} - \frac{y_c^4}{48} + \frac{y_c^3}{72}\right)\right\} dy_c$$

$$= \frac{K_1 H}{L} + \frac{2K_2}{\delta} - \frac{1}{3840\delta}\left(\frac{K_1 H}{L}\right)^2 Ra_H$$

5. $$q' \equiv k \int_{-H/2}^{H/2}\left(-\frac{\partial T}{\partial x}\right)_{x=0} dy = k\Delta T \int_{-1/2}^{1/2}\left(-\frac{\partial T_e}{\partial x_1}\right)_{x_1=0} dy_c$$

6. $$\overline{Nu} \equiv \frac{q'}{(kH\Delta T)/L} = \frac{k\Delta T\left\{\frac{K_1 H}{L} + \frac{2K_2}{\delta} - \frac{1}{3840\delta}\left(\frac{K_1 H}{L}\right)^2 Ra_H\right\}}{(kH\Delta T)/L}$$

$$= K_1 + \frac{2L}{\delta H}\left\{K_2 - \frac{1}{7680}\left(\frac{K_1 H}{L}\right)^2 Ra_H\right\}$$

$$= K_1 + \frac{2L}{\delta H}\left\{\frac{1-K_1}{2} - \left(\frac{K_1 H}{L}\right)^2 \frac{Ra_H}{1440} - \frac{1}{7680}\left(\frac{K_1 H}{L}\right)^2 Ra_H\right\}$$

$$= K_1 + \frac{2L}{\delta H}\left\{\frac{1-K_1}{2} - \frac{19}{23040}\left(\frac{K_1 H}{L}\right)^2 Ra_H\right\}$$

$$= K_1 + 2\left\{\left(\frac{H}{L}\right)^2 K_1^3 \frac{Ra_H^2}{725760}\right\}\left\{1 - \frac{38}{23040}\frac{1}{1-K_1}\left(\frac{K_1 H}{L}\right)^2 Ra_H\right\}$$

$$\cong K_1 + \frac{K_1^3}{362880}\left(\frac{H}{L} Ra_H^2\right)^2$$