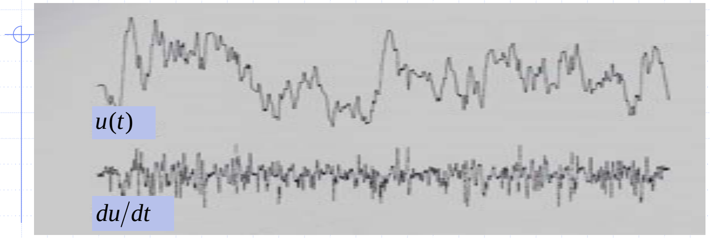
5 Space and Time Scales of Turbulence



- largest time scale = the time interval for statistical decorrelation
- stop fluctuating (signals become smooth) until the smallest time scale ~ τ_η is reached
- self-similar: fluctuates if examined at successively magnifications

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5 Space and Time Scales of Turbulence

$\frac{3}{2}q^2$ turbulent intensity

$$\vec{u} = <\vec{u}> + \vec{u}'$$

$$\left\langle \tfrac{1}{2} u_i u_i \right\rangle = \underbrace{\tfrac{1}{2} < u_i >< u_i >}_{\text{energy of mean motion}} + \underbrace{\tfrac{1}{2} < u_i' u_i' >}_{\text{turbulent energy}}$$

largest time scale ~ $T = \int_0^\infty \frac{< u'(\vec{x},t) u'(\vec{x},t+\tau) >}{\sqrt{\left\langle u'(t)^2 \right\rangle \left\langle u'(t+\tau)^2 \right\rangle}} d\tau$

smallest time scale ~ $\tau_\eta = (\nu/\varepsilon)^{1/2}$

largest length scale ~ $L = \int_0^\infty \frac{< u'(\vec{x},t) u'(\vec{x}+\vec{r},t) >}{\sqrt{\left\langle u'(\vec{x})^2 \right\rangle \left\langle u'(\vec{x}+\vec{r})^2 \right\rangle}} dr$

smallest length scale ~ $\eta = \left(\nu^3/\varepsilon\right)^{1/4}$

Turbulence contains a continuum of scales ranging from the large ones (~ T,L) to small ones (~ τ_η,η).

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5 Space and Time Scales of Turbulence

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial \frac{1}{2} u_i u_i}{\partial t} + u_j \frac{\partial \frac{1}{2} u_i u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial u_i p}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(2 u_i S_{ij}\right) - \Delta$$

$$\Delta = \tfrac{1}{2}\nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 2\nu S_{ij} S_{ij}$$

The viscous dissipation grows as the scale decreases.

Energy Cascade: As stretched and folded by convection, large eddies progressively develop small eddies through their evolution during a time of L/q (energy is cascaded from large scales to small scales).

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5 Space and Time Scales of Turbulence

Energy Cascade

- Energy is cascaded from large scales to small scales through through stretching and folding by convection.
- Evolution of large eddies controls the rate at which energy is fed through to be dissipated.
- Turbulence decides its own smallest scales according to the viscosity and the energy cascade rate.
- When steady, the small eddies dissipate energy at a rate equal to the cascade rate.
- The instabilities of the mean flow replenish the large scales of turbulence.

5 Space and Time Scales of Turbulence

Taylor Microscale

assume isotropic homogeneous turbulence

two-point-one-time velocity correlation coefficient:

$$\rho_{ij}(\vec{r},t) = <u_i'(\vec{x},t)u'_j(\vec{x}+\vec{r},t)>/q^2$$

$$q = \sqrt{<u_1'u_1'>} = \sqrt{<u_2'u_2'>} = \sqrt{<u_3'u_3'>}$$

$\rho_{11}(\vec{r},t), \rho_{22}(\vec{r},t), \rho_{33}(\vec{r},t)$: even functions

$$\rho_{11}(r_1 = r, r_2 = r_3 = 0) \begin{cases} \rho_{11}(r_1 = 0) = 1 \\ \rho_{11}(r_1 \to \infty) = 0 \\ \rho_{11}(r_1) \text{ significantly less than one when} \\ \qquad r \sim L = \int_0^\infty \rho_{11}(r_1)dr_1 \end{cases}$$

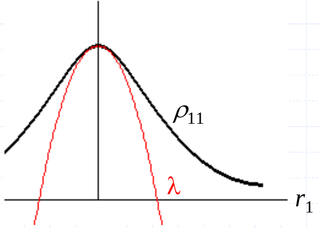
5 Space and Time Scales of Turbulence

Taylor Microscale λ

$$\rho_{11}(\vec{r},t) = <u_1'(\vec{x},t)u_1'(\vec{x}+r_1\vec{e}_1,t)>/q^2$$

$$\rho_{11}(r_1) = 1 + 0 + \frac{1}{2}\left(\frac{\partial^2\rho_{11}}{\partial r^2}\right)_{r=0} r_1^2 + 0 + \cdots$$

Taylor's series about $r = 0$: $\rho_{11}(r_1) = 1 - \frac{r_1^2}{\lambda^2} + \cdots$



5 Space and Time Scales of Turbulence

Taylor Microscale λ : $\rho_{11}(r_1) = 1 - \frac{r_1^2}{\lambda^2} + \cdots$

Althernatively, $R_{11}(r_1) = <u_1'(x)u_1'(x+r_1)> = \rho_{11}(r_1)\cdot q^2$

$$\frac{\partial}{\partial r_1}R_{11}(r_1) = <u_1'(x)\frac{\partial}{\partial r_1}u_1'(x+r_1)> = <u_1'(x)\frac{\partial u_1'(x+r_1)}{\partial(x+r_1)}>$$

$$= <u_1'(x)\frac{\partial u_1'}{\partial x}(x+r_1)> = <u_1'(x-r_1)\frac{\partial u_1'}{\partial x}(x)>$$

$$\frac{\partial^2 R_{11}}{\partial r_1^2}(r_1) = \frac{\partial}{\partial r_1}<u_1'(x-r_1)\frac{\partial u_1'}{\partial x}(x)> = <-\frac{\partial u_1'(x-r_1)}{\partial(x-r_1)}\cdot\frac{\partial u_1'}{\partial x}(x)>$$

$$\frac{\partial^2 R_{11}}{\partial r_1^2}(r_1) = -<\frac{\partial u_1'}{\partial x}(x-r_1)\cdot\frac{\partial u_1'}{\partial x}(x)>$$

$$\Rightarrow \frac{\partial^2 R_{11}}{\partial r_1^2}(0) = -<\frac{\partial u_1'}{\partial x}\frac{\partial u_1'}{\partial x}(x)> = -\frac{2q^2}{\lambda^2}$$

5 Space and Time Scales of Turbulence

Taylor Microscale λ : $<\left(\frac{\partial u_1'}{\partial x}\right)^2> = \frac{2q^2}{\lambda^2}$

Consider the order of magnitudes: $<\left|\frac{\partial u_i'}{\partial x_j}\right|> \sim \frac{q}{\lambda}$

$$\bar{\varepsilon} = \tfrac{1}{2}\nu<\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)> \sim \frac{\nu q^2}{\lambda^2}$$

Recall $\eta = \left(\nu^3/\bar{\varepsilon}\right)^{1/4}$ and $L/\eta = \mathrm{Re}_L^{3/4}$

$$\frac{\lambda}{\eta} \sim \frac{qL}{\nu}\cdot\frac{\eta}{L} \sim \mathrm{Re}_L^{1/4} \quad \text{or} \quad \frac{\lambda}{L} \sim \mathrm{Re}_L^{-1/2}$$

When Re is sufficiently high, $\eta << \lambda << L$

5 Space and Time Scales of Turbulence

Reynolds number

$$\mathrm{Re}_L \equiv \frac{qL}{\nu}$$ ~ a measure of the significance of viscosity for the large scales of turbulence

$$\mathrm{Re}_\lambda \equiv \frac{q\lambda}{\nu}$$ ~ Taylor Reynolds number

$$\frac{\lambda}{\eta} \sim \frac{qL}{\nu}\cdot\frac{\eta}{L} \sim \mathrm{Re}_L^{1/4} \Longrightarrow \boxed{\mathrm{Re}_L = \mathrm{Re}_\lambda^2}$$

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5 Space and Time Scales of Turbulence

one-point-two-time correlation coefficients

Steady: $\rho_{ij}(\vec{x},\tau) = < u_i'(\vec{x},t)u_j'(\vec{x},t+\tau) > / q^2$

$$\begin{cases} \rho_{11}(\tau=0)=1 \\ \rho_{11}(\tau\to\infty)=0 \\ \rho_{11}(\tau) \text{ significantly less than one when} \\ \qquad \tau \sim T = \int_0^\infty \rho_{11}(\tau)d\tau \sim L/q \end{cases}$$

~ a measure of Eulerian time scales

Energy contained in large eddies ~ q^2 is cascaded down to smaller eddies in a time ~ L/q. Therefore the energy cascade rate (=dissipation rate since steady) is

$$\bar{\varepsilon} \sim q^3/L \quad \text{or} \quad L \sim q^3/\bar{\varepsilon}$$

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5 Space and Time Scales of Turbulence

Eddy lifetime v.s. Eulerian time scale

velocity difference across a distance $\ell = \Delta u_\ell = < u'(x+\ell) - u'(x) >$

eddy lifetime of size $\ell = \ell/\Delta u_\ell$

Eulerian time scale is dominated by the sweeping of mean flow/large eddies past a fixed point

$= \ell/q$ (if the frame moves with the mean flow)

$$\frac{\text{lifetime}}{\text{Eulerian time}} \sim \frac{q}{\Delta u_\ell} >> 1 \text{ for small enough } \ell$$

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6. Mean Motion

~ one-point-one-time statistics
~ essential and important but not complete

§ Reynolds (1894) Averaged Velocity

$u_i = \bar{u}_i + u_i'$ = mean + turbulent velocity

mean kinetic energy per unit mass = $\frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\bar{u}_i\bar{u}_i + \frac{1}{2}\overline{u_i' u_i'} \equiv \bar{K} + K$

= energy of mean motion + turbulent energy

(a) Continuity: $\nabla\cdot\vec{u} = \frac{\partial u_j}{\partial x_j} = \frac{\partial(\bar{u}_j + u_j')}{\partial x_j} = 0$ (1) $\Longrightarrow \overline{\frac{\partial u_j}{\partial x_j}} = \frac{\partial \bar{u}_j}{\partial x_j} = 0$ (1a)

(1)-(1a) $\Longrightarrow \frac{\partial u_j'}{\partial x_j} = 0$ (1b)

6. Mean Motion

§ Reynolds (1894) Averaged Navier-Stokes Equations (RANS)

(b) Momentum:

$$\frac{\partial u_i}{\partial t}+\frac{\partial\left(u_i u_j\right)}{\partial x_j}=g_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\mu}{\rho}\frac{\partial \tau_{ji}}{\partial x_j} \qquad (2)$$

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

$u_i=\bar{u}_i+u_i'$ ⇒

$$\frac{D\bar{u}_i}{Dt}\equiv\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}=g_i-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ji}-\rho\overline{u_i'u_j'}\right) \qquad (2a)$$

$$\bar{\tau}_{ij}=\mu\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)$$

mean viscous
stress tensor

6. Mean Motion

$\tau_{ij}^t=-\rho\,\overline{u_i'\,u_j'}$ = Reynolds (turbulent) stress tensor

- ~ the average momentum flux due to turbulent velocity fluctuations
- ~ the interaction (coupling) of turbulence with the mean flow
- ~ arising from the nonlinear (convection) term of Navier-Stokes equations
- ~ cause the closure problem
- ~ much larger than viscous stress except near very walls where $\frac{\partial\bar{u}_i}{\partial x_j}$ is not small for generally large-Reynolds-number turbulent flows

(As μ→0, $\bar{\tau}_{ij}=\mu\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\rightarrow 0$ because $\bar{u}_i$ does not fluctuate.)

- ~ homogeneous turbulence has no effect on the mean flow, $\frac{\partial\rho\overline{u_i'u_j'}}{\partial x_j}=0$

6. Mean Motion

Turbulent momentum equations: total momentum – mean momentum

$$\frac{Du_i'}{Dt}\equiv\frac{\partial u_i'}{\partial t}+\bar{u}_j\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{1}{\rho}\frac{\partial \tau'_{ji}}{\partial x_j} \qquad (2b)$$

$$\tau'_{ij}=\mu\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)+\rho\left(\overline{u_i'u'_j}-\bar{u}_i u'_j-u_i'u'_j\right)$$

(c) Energy of mean motion = $\bar{K}\equiv\bar{u}_i\bar{u}_i/2$

$$\bar{u}_i\cdot\left\{\rho\frac{D\bar{u}_i}{Dt}=\rho g_i-\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ji}-\rho\overline{u_i'u_j'}\right)\right\}$$

$$\rho\frac{D\bar{K}}{Dt}=\rho\bar{u}_i g_i-\bar{u}_i\frac{\partial\bar{p}}{\partial x_i}+\bar{u}_i\frac{\partial}{\partial x_j}\mu\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)-\bar{u}_i\frac{\partial\rho\overline{u_i'u'_j}}{\partial x_j}$$

6. Mean Motion

Energy of mean motion

pressure work

turbulent transport

$$\rho\frac{D\bar{K}}{Dt}=\rho\bar{u}_i g_i-\frac{\partial\left(\bar{u}_j\bar{p}\right)}{\partial x_j}+\frac{\partial}{\partial x_j}\left\{\mu\bar{u}_i\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)-\bar{u}_i\rho\overline{u_i'u_j'}\right\}$$

body force work

viscous diffusion

$$\boxed{-2\mu\bar{S}_{ij}\bar{S}_{ij}}\quad+\quad\rho\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}$$

negative
mostly

molecular dissipation
always negative

turbulent cascade

$$\bar{S}_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)$$

6. Mean Motion

Energy of mean motion (no body force)

$$\rho\frac{D\overline{K}}{Dt}=-2\mu\overline{S}_{ij}\overline{S}_{ij}\quad+\quad\rho\overline{u_i'u_j'}\frac{\partial\overline{u}_i}{\partial x_j}$$

$$+\frac{\partial}{\partial x_j}\left\{-\overline{u}_j\overline{p}+\mu\overline{u}_i\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right)-\overline{u}_i\rho\overline{u_i'u_j'}\right\}$$

~ diffusion due to inhomogeneities

~ vanish when integrate over the whole flow domain

~ vanish in homogeneous turbulence

$$\rho\frac{D\overline{K}}{Dt}=\underbrace{-2\mu\overline{S}_{ij}\overline{S}_{ij}}_{\substack{\text{viscous}\\\text{dissipation}\\\text{(irreversible)}}}-\underbrace{\left(-\rho\overline{u_i'u_j'}\frac{\partial\overline{u}_i}{\partial x_j}\right)}_{\substack{\text{energy cascade rate}\\\text{(reversible)}}}$$

6. Mean Motion

Turbulent Kinetic Energy $K\equiv\overline{u_i'u_i'}/2=3q^2/2$

(d) Turbulent Kinetic Energy:

$$\overline{u_i'\left\{\frac{Du_i'}{Dt}\equiv\frac{\partial u_i'}{\partial t}+\overline{u}_j\frac{\partial u_i'}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{1}{\rho}\frac{\partial\tau'_{ji}}{\partial x_j}\right\}}$$

convection by mean motion

turbulent transport

$$S_{ij}'=\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)$$

viscous diffusion

viscous dissipation

$$\rho\frac{DK}{Dt}=\rho\frac{\partial K}{\partial t}+\rho\overline{u}_j\frac{\partial K}{\partial x_j}$$

$$=\underbrace{\frac{\partial}{\partial x_j}\left\{-\overline{p'u'_j}-\tfrac{1}{2}\rho\overline{u_i'u_i'u'_j}+\mu\frac{\partial K}{\partial x_j}\right\}}_{\text{diffusion due to inhomogeneities}}\underbrace{-\rho\overline{u_i'u'_j}\frac{\partial\overline{u}_i}{\partial x_j}}_{\substack{\text{turbulent}\\\text{production/destruction}\\\text{strong in regions with}\\\text{large mean shear}}}-2\mu\overline{S_{ij}'S_{ij}'}$$

6. Mean Motion

energy dissipation rate per unit mass

mean flow dissipation rate: $\Sigma=2\nu\overline{S}_{ij}\overline{S}_{ij}$, $\overline{S}_{ij}=\frac{1}{2}\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right)$

mean turbulent dissipation rate: $\overline{\varepsilon}=2\nu\overline{S_{ij}'S_{ij}'}$, $S_{ij}'=\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)$

total dissipation rate: $\overline{\Delta}=\Sigma+\overline{\varepsilon}=2\nu\left(\overline{S}_{ij}\overline{S}_{ij}+\overline{S_{ij}'S_{ij}'}\right)$

~ intermittent ε and Δ (local/one ensemble turbulent and total dissipation rate)

~ At high Reynolds numbers, usually $\overline{\varepsilon}>>\Sigma$ (dissipation dominated by small scales)

(The characteristic scales for a not-small variation of mean quantity is usually comparable or larger than the largest scales of turbulence, except near the walls.)

6. Mean Motion

energy dissipation rate per unit mass

mean turbulent dissipation rate:

$$\overline{\varepsilon}=2\nu\overline{S_{ij}'S_{ij}'}=\frac{1}{2}\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}$$

$$\boxed{\begin{array}{l}\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\sim\frac{\overline{u_i'u_i'}}{\lambda^2}\\[2ex]\frac{\partial^2\overline{u_i'u_i'}}{\partial x_m\partial x_m}\sim\frac{\overline{u_i'u_i'}}{L^2}<<\frac{\overline{u_i'u_i'}}{\lambda^2}\end{array}}$$

$$=\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}+\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u'_j}{\partial x_i}}$$

$$=\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}+\nu\frac{\partial^2\overline{u_i'u'_j}}{\partial x_i\partial x_j}\approx\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

$$\boxed{\omega_i=\varepsilon_{irs}\frac{\partial u_s}{\partial x_r}}$$

$$=\nu\overline{\omega_i'\omega_i'}+2\nu\frac{\partial^2\overline{u_i'u'_j}}{\partial x_i\partial x_j}$$

$$\approx\nu\overline{\omega_i'\omega_i'}$$

6. Mean Motion

Turbulent decay

Without no mean velocity gradients, there is no turbulent production.

$$-\rho\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}=0$$

⇨ Turbulent energy decays continuously.

time scale for turbulent decay $\sim \frac{K}{\varepsilon}=\frac{\frac{3}{2}q^2}{\varepsilon}\sim\frac{q^2}{\varepsilon}$

time scale of energy supply from the large scales for dissipation at small scales $\sim\frac{L}{q}$

⇨ $\varepsilon\sim\frac{q^3}{L}$ or $L\sim\frac{q^3}{\varepsilon}$

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6. Mean Motion

Reynolds stress tensor equations

$$u'_j\left\{\frac{Du_i'}{Dt}=\frac{\partial u_i'}{\partial t}+\bar{u}_m\frac{\partial u_i'}{\partial x_m}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\frac{1}{\rho}\frac{\partial \tau'_{mi}}{\partial x_m}\right\}$$

$$+\ )\qquad u_i'\left\{\frac{Du'_j}{Dt}=\frac{\partial u'_j}{\partial t}+\bar{u}_m\frac{\partial u'_j}{\partial x_m}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_j}+\frac{1}{\rho}\frac{\partial \tau'_{mj}}{\partial x_m}\right\}$$

$$\frac{Du_i'u'_j}{Dt}=\frac{\partial u_i'u'_j}{\partial t}+\bar{u}_m\frac{\partial\left(u_i'u'_j\right)}{\partial x_m}=-\frac{1}{\rho}\left(u'_j\frac{\partial p'}{\partial x_i}+u_i'\frac{\partial p'}{\partial x_j}\right)+\frac{1}{\rho}\left(u'_j\frac{\partial \tau'_{mi}}{\partial x_m}+u_i'\frac{\partial \tau'_{mj}}{\partial x_m}\right)$$

$$\overline{u'_j\frac{\partial \tau'_{mi}}{\partial x_m}}=\overline{u'_j\frac{\partial}{\partial x_m}\left\{\mu\left(\frac{\partial u_i'}{\partial x_m}+\frac{\partial u'_m}{\partial x_i}\right)+\rho\left(\overline{u_i'u_m'}-\bar{u}_iu_m'-u_i'u_m'\right)\right\}}$$

$$=\mu\overline{u'_j\frac{\partial^2u_i'}{\partial x_m\partial x_m}}-\rho\frac{\partial\bar{u}_i}{\partial x_m}\cdot\overline{u'_ju_m'}-\rho\overline{\frac{\partial u_i'}{\partial x_m}u'_ju_m'}$$

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6. Mean Motion

Reynolds stress tensor equations

$$\frac{D\overline{u_i'u'_j}}{Dt}=\frac{\partial\overline{u_i'u'_j}}{\partial t}+\bar{u}_m\frac{\partial\overline{u_i'u'_j}}{\partial x_m}$$ ~ mean motion Lagrangian

→ $$=-\frac{1}{\rho}\overline{\left(u'_j\frac{\partial p'}{\partial x_i}+u_i'\frac{\partial p'}{\partial x_j}\right)}$$ ~ pressure effects (nonlocal, linear, and nonlinear)

$$+\nu\left(\frac{\partial^2\overline{u_i'u'_j}}{\partial x_m\partial x_m}-2\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}\right)$$ ~ viscous diffusion/dissipation effect

nonlocal

$$-\left(\overline{u_i'u_m'}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju_m'}\frac{\partial\bar{u}_i}{\partial x_m}\right)$$ ~ production and reorientation by the mean motion

→ $$-\frac{\partial\overline{u_i'u'_ju_m'}}{\partial x_m}$$ ~ turbulent advection

closure problem

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6. Mean Motion

Reynolds stress tensor equations for homogeneous turbulence

$$\frac{D\overline{u_i'u'_j}}{Dt}=-\frac{1}{\rho}\overline{p'\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right)}-2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}-\left(\overline{u_i'u_m'}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju_m'}\frac{\partial\bar{u}_i}{\partial x_m}\right)$$

- isotropic turbulence: $\overline{u_i'u'_j}=\frac{1}{3}\overline{u_k'u_k'}\delta_{ij}$
- Anisotropy arises from pressure-strain correlations, viscous effect, and mean-flow gradients,
- Pressure fluctuations redistribute turbulent energy only in different directions.

6. Effect of Pressure

conservative

$$\frac{\partial}{\partial x_i}\left\{ \frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \cancel{g_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial \tau_{ji}}{\partial x_j} \right\}$$

$$\frac{\partial \tau_{ij}}{\partial x_i \partial x_j} = \mu \frac{\partial}{\partial x_i \partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 (u_i u_j)}{\partial x_i \partial x_j} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i}$$

$$\nabla^2 p = -\frac{\partial^2 (\rho u_i u_j)}{\partial x_i \partial x_j}$$

~ Poisson equation for pressure

6. Effect of Pressure

$$\nabla^2 p = -\frac{\partial^2 (\rho u_i u_j)}{\partial x_i \partial x_j}$$

Neumann type BC : $\left\{ \frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial \tau_{ji}}{\partial x_j} \right\} \cdot n_i$

$$\nabla^2 \overline{p} = -\frac{\partial^2 (\rho \overline{u_i u_j})}{\partial x_i \partial x_j} = -\frac{\partial^2}{\partial x_i \partial x_j}\left(\rho \overline{u}_i \overline{u}_j + \rho \overline{u'_i u'_j} \right)$$

$$\nabla^2 p' = -\frac{\partial^2}{\partial x_i \partial x_j}\left(\rho u'_i \overline{u}_j + \rho \overline{u}_i u'_j + \rho u'_i u'_j - \rho \overline{u'_i u'_j} \right)$$

~ Poisson equations for mean pressure and pressure fluctucation

6. Effect of Pressure

unbounded (infinite) flows

$$\overline{p}(\vec{x}) = \frac{\rho}{4\pi}\iiint \frac{\partial^2}{\partial x'_i \partial x'_j}\left(\overline{u}_i(\vec{x}')\overline{u}_j(\vec{x}') + \overline{u'_i u'_j}(\vec{x}') \right)\frac{d\vec{x}'}{|\vec{x} - \vec{x}'|}$$

nonlocal and nonlinear

$$p'(\vec{x}) = \frac{\rho}{4\pi}\iiint \frac{\partial^2}{\partial x'_i \partial x'_j}\left\{ u'_i \overline{u}_j + \overline{u}_i u'_j + u'_i u'_j - \overline{u'_i u'_j} \right\}(\vec{x}')\frac{d\vec{x}'}{|\vec{x} - \vec{x}'|}$$

$$\boxed{\overline{p'(\vec{x})u'_k(\vec{x})}} \quad \overline{u'_k(\vec{x})\left\{ u'_i(\vec{x}')\overline{u}_j(\vec{x}') + \overline{u}_i(\vec{x}')u'_j(\vec{x}') + u'_i(\vec{x}')u'_j(\vec{x}') - \cancel{\overline{u'_i u'_j}(\vec{x}')} \right\}}$$

$$\overline{p'u'_k}(\vec{x}) = \frac{\rho}{4\pi}\iiint \frac{\partial^2}{\partial x'_i \partial x'_j}\left\{ \begin{array}{l} \overline{u}_j(\vec{x}')\overline{u'_i(\vec{x}')u'_k(\vec{x})} \\ + \overline{u}_i(\vec{x}')\overline{u'_j(\vec{x}')u'_k(\vec{x})} + \overline{u'_i(\vec{x}')u'_j(\vec{x}')u'_k(\vec{x})} \end{array} \right\}\frac{d\vec{x}'}{|\vec{x} - \vec{x}'|}$$

one-point pressure-velocity correlation in terms of double and triple velocity correlations at two points ⇨ **closure+nonlocalness**

6. Effect of Pressure

$$\nabla^2\left\{ p'^{(L)} + p'^{(NL)} \right\} = -\frac{\partial^2}{\partial x_i \partial x_j}\left\{ \underbrace{\rho u'_i \overline{u}_j + \rho \overline{u}_i u'_j}_{\text{linear}} + \underbrace{\rho u'_i u'_j - \rho \overline{u'_i u'_j}}_{\text{nonlinear}} \right\}$$

- ❑ assuming homogeneous turbulence, $-\frac{\partial \overline{p'u'_j}}{\partial x_j} = 0$, the pressure fluctuations redistribute turbulent energy only among different directions

- ❑ $\overline{p'^{(NL)}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)} = 0$ in isotropic turbulence in the absence of boundaries, because $p'^{(NL)}$ does not depend on the mean flow and the flow is incompressible.

6. Examples

channel flow in between two parallel solid walls

$$\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)-\rho\overline{u'_iu'_j}\right)$$

Assumptions: steady, fully developed, 2D (mean motion)

$$\frac{\partial\overline{(\cdot)}}{\partial x_1}=\frac{\partial\overline{(\cdot)}}{\partial x_3}=0$$

$$\bar{u}_2=\bar{u}_3=0$$

$$i=1:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_1}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(\mu\frac{\partial\bar{u}_1}{\partial x_2}-\rho\overline{u'_1u'_2}\right)$$

$$i=2:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_2}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(-\rho\overline{u'^2_2}\right)$$

$$i=3:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_3}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(-\rho\overline{u'_2u'_3}\right)$$

6. Examples

Expect: symmetry about $x_3 = 0$ plane $\Rightarrow$ $\overline{u'_1u'_3}=\overline{u'_2u'_3}=0$

$$i=1:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_1}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(\mu\frac{\partial\bar{u}_1}{\partial x_2}-\rho\overline{u'_1u'_2}\right)\Rightarrow\frac{d\bar{p}_w}{dx_1}=\frac{d}{dx_2}\left(\mu\frac{d\bar{u}_1}{dx_2}-\rho\overline{u'_1u'_2}\right)$$

$$i=2:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_2}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(-\rho\overline{u'^2_2}\right)\Rightarrow\bar{p}(x_1,x_2)=-\rho\overline{u'^2_2}(x_2)+\bar{p}_w(x_1)$$

$$i=3:\quad 0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_3}+\frac{1}{\rho}\frac{\partial}{\partial x_2}\left(-\rho\cancel{\overline{u'_2u'_3}}\right)$$

$$\Rightarrow\bar{\tau}(x_2)\equiv\mu\frac{d\bar{u}_1}{dx_2}-\rho\overline{u'_1u'_2}=\text{total stress}=\frac{d\bar{p}_w}{dx_1}x_2+\bar{\tau}_w$$

At $x_2 = D$, $\frac{d\bar{u}_1}{dx_2}=\rho\overline{u'_1u'_2}=0$: $0=\frac{d\bar{p}_w}{dx_1}D+\bar{\tau}_w\Rightarrow\frac{\bar{\tau}(x_2)}{\bar{\tau}_w}=\left(1-\frac{x_2}{D}\right)$

6. Examples

Turbulent energy

$$\rho\frac{DK}{Dt}=\frac{\partial}{\partial x_j}\left\{-\overline{p'u'_j}-\tfrac{1}{2}\rho\overline{u'_iu'_iu'_j}+\mu\frac{\partial K}{\partial x_j}+\mu\frac{\partial\overline{u'_iu'_j}}{\partial x_i}\right\}-\rho\overline{u'_iu'_j}\frac{\partial\bar{u}_i}{\partial x_j}-2\mu\overline{S'_{ij}S'_{ij}}$$

$$\Downarrow$$

$$0=\frac{\partial}{\partial x_2}\left\{-\overline{p'u'_2}-\tfrac{1}{2}\rho\overline{u'_iu'_iu'_2}+\mu\frac{\partial K}{\partial x_2}+\mu\frac{\partial\overline{u'_iu'_2}}{\partial x_i}\right\}-\rho\overline{u'_iu'_2}\frac{\partial\bar{u}_i}{\partial x_2}-2\mu\overline{S'_{ij}S'_{ij}}$$

$$\Downarrow$$

$$0=\frac{\partial}{\partial x_2}\left\{-\overline{p'u'_2}-\tfrac{1}{2}\rho\overline{u'_iu'_iu'_2}+\mu\frac{\partial}{\partial x_2}\left(K+\overline{u'^2_2}\right)\right\}-\rho\overline{u'_1u'_2}\frac{\partial\bar{u}_1}{\partial x_2}-2\mu\overline{S'_{ij}S'_{ij}}$$

turbulent energy transfer across the channel by the x_2-component of turbulent velocity

rate of turbulence production

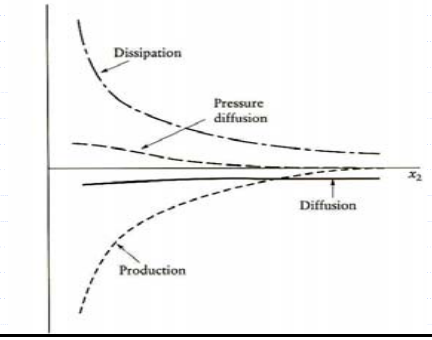
nonzero due to inhomogeneity in x_2 direction

6. Examples

$$\int_0^{2D}\left\{0=\frac{\partial}{\partial x_2}\left\{-\overline{p'u'_2}-\tfrac{1}{2}\rho\overline{u'_iu'_iu'_2}+\mu\frac{\partial}{\partial x_2}\left(K+\overline{u'^2_2}\right)\right\}-\rho\overline{u'_1u'_2}\frac{\partial\bar{u}_1}{\partial x_2}-2\mu\overline{S'_{ij}S'_{ij}}\right\}dx_2$$

$$\Downarrow$$

$$-\int_0^{2D}\rho\overline{u'_1u'_2}\frac{\partial\bar{u}_1}{\partial x_2}dx_2=2\mu\int_0^{2D}\overline{S'_{ij}S'_{ij}}dx_2=\int_0^{2D}\bar{\varepsilon}dx_2$$



6. Examples

steady, infinite mean flow with uniform shear

$$\bar{u}_1 = sx_2, \quad \bar{u}_2 = \bar{u}_3 = 0$$

- a solution with constant mean pressure

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \rho\overline{u'_i u'_j}\right)$$

- Initially homogeneous turbulence will remain so.

$$\frac{Du'_i}{Dt} \equiv \frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} + u'_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u'_i}{\partial x_j} + \overline{u'_i u'_j} - u'_i u'_j\right)$$

- turbulent energy

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = -\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - 2\nu\overline{S'_{ij} S'_{ij}} \quad \text{(homogeneous)}$$

$$\frac{\partial K}{\partial t} + \bar{u}_1 \frac{\partial K}{\partial x_1} = -s\overline{u'_1 u'_2} - \bar{\varepsilon}$$

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6. Examples

steady, infinite mean flow with uniform shear

$$\bar{u}_1 = sx_2, \quad \bar{u}_2 = \bar{u}_3 = 0$$

- Reynolds stresses

$$\overline{u'_1 u'_3} = \overline{u'_2 u'_3} = 0 \quad \text{due to symmetry}$$

$$\frac{D\overline{u'_1 u'_2}}{Dt} = -\frac{1}{\rho}\overline{p'\left(\frac{\partial u'_1}{\partial x_2} + \frac{\partial u'_2}{\partial x_1}\right)} - 2\nu\overline{\frac{\partial u'_1}{\partial x_m}\frac{\partial u'_2}{\partial x_m}} - s\overline{u'^2_2}$$

- turbulent pressure

$$\nabla^2\left\{p'^{(L)} + p'^{(NL)}\right\} = -\frac{\partial^2}{\partial x_i \partial x_j}\left\{\rho u'_i \bar{u}_j + \rho \bar{u}_i u'_j + \rho u'_i u'_j - \cancel{\rho\overline{u'_i u'_j}}\right\}$$

$$\nabla^2\left\{p'^{(L)} + p'^{(NL)}\right\} = -2\rho s \frac{\partial u'_2}{\partial x_1} - \rho \frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}$$

$$p'^{(L)} \equiv s\Pi$$

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6. Examples

steady, infinite mean flow with uniform shear

$$\bar{u}_1 = sx_2, \quad \bar{u}_2 = \bar{u}_3 = 0$$

- isotropic turbulence

$$\overline{u'^2_1} = \overline{u'^2_2} = \overline{u'^2_3} = q^2$$

$$\overline{u'_i u'_j} = q^2 \delta_{ij}$$

- a measure of anisotropy

$$\frac{D\overline{u'_1 u'_2}}{Dt} = -s\overline{u'^2_2} - \frac{s}{\rho}\overline{\Pi\left(\frac{\partial u'_1}{\partial x_2} + \frac{\partial u'_2}{\partial x_1}\right)} - \frac{1}{\rho}\overline{p'^{(NL)}\left(\frac{\partial u'_1}{\partial x_2} + \frac{\partial u'_2}{\partial x_1}\right)} - 2\nu\overline{\frac{\partial u'_1}{\partial x_m}\frac{\partial u'_2}{\partial x_m}}$$

effects of mean shear on turbulence

tend to resist the growth of $\overline{u'_1 u'_2}$ against $-s\overline{u'^2_2}$