Randomness ⇨ probability and statistics

Velocity = mean + fluctuation (turbulent)

$$\vec{u} = <\vec{u}> + \vec{u}'$$

probability distribution function $P(\vec{u})$

$P(\vec{u})d\vec{u}$ = probability of finding the velocity $\in (\vec{u}, \vec{u}+d\vec{u})$ at a given location and a given time

$$\int\int\int_{-\infty}^{\infty} P(\vec{u})d\vec{u} = 1$$

$$<\vec{u}> = \int\int\int_{-\infty}^{\infty} \vec{u}P(\vec{u})d\vec{u}$$

$$\vec{u}' = \vec{u} - <\vec{u}>$$

P.S. It is formally defined via ensemble of experiments.

Statistical Moments: $<u^\gamma> = \int_{-\infty}^{\infty} u^\gamma P(u)du$

Central moments:

$$\mu_\gamma \equiv <(u-<u>)^\gamma> = \int_{-\infty}^{\infty} (u-<u>)^\gamma P(u)du$$

$$\mu_1 = 0$$

$$\mu_2 = \sigma^2 = \text{variance of } u$$

: measure how far about its mean u varies

$$S \equiv \frac{\mu_3}{\sigma^3} = \text{skewness of } u$$

: measure of lack of symmetry of $P(u)$

$$K \equiv \frac{\mu_4}{\sigma^4} = \text{flatness factor (kurtosis) of } u$$

: measure how extensive the tails of $P(u)$ are

$$<f(\vec{u})> = \int\int\int_{-\infty}^{\infty} f(\vec{u})P(\vec{u})d\vec{u}$$

$$<f(\vec{u}) + g(\vec{u})> = <f(\vec{u})> + <g(\vec{u})>$$

$$<\lambda f(\vec{u})> = \lambda <f(\vec{u})>$$ λ : constant

$$<\frac{\partial \vec{u}}{\partial x}> = \frac{\partial <\vec{u}>}{\partial x}$$

$$= <\lim_{h\to 0} \frac{\vec{u}(x+h) - \vec{u}(x)}{h}> = \lim_{h\to 0} \frac{<\vec{u}(x+h)> - <\vec{u}(x)>}{h}$$

Joint probability distribution function $P(\vec{u}_1, \vec{u}_2)$

$P(\vec{u}_1, \vec{u}_2)d\vec{u}_1 d\vec{u}_2$

= probability of finding velocity $\in (\vec{u}_1, \vec{u}_1 + d\vec{u}_1)$ at $(\vec{x}_1, t_1)$

and finding velocity $\in (\vec{u}_2, \vec{u}_2 + d\vec{u}_2)$ at $(\vec{x}_2, t_2)$

$$\int\int\int_{-\infty}^{\infty} d\vec{u}_2 \int\int\int_{-\infty}^{\infty} d\vec{u}_1 P(\vec{u}_1, \vec{u}_2) = 1$$

$$P(\vec{u}_1) = \int\int\int_{-\infty}^{\infty} P(\vec{u}_1, \vec{u}_2)d\vec{u}_2$$

$$<f(\vec{u}_1, \vec{u}_2)> = \int\int\int_{-\infty}^{\infty} d\vec{u}_1 \int\int\int_{-\infty}^{\infty} d\vec{u}_2 f(\vec{u}_1, \vec{u}_2)P(\vec{u}_1, \vec{u}_2)$$

Conditional probability distribution function $P(\vec{u}_1 \mid \vec{u}_2)$

$P(\vec{u}_1 \mid \vec{u}_2) d\vec{u}_1$

= probability of finding velocity $\in (\vec{u}_1, \vec{u}_1 + d\vec{u}_1)$ at $(\vec{x}_1, t_1)$

conditional on having velocity $\in (\vec{u}_2, \vec{u}_2 + d\vec{u}_2)$ at $(\vec{x}_2, t_2)$

$$= \frac{\text{ensembles satisfying both conditions}}{\text{ensembles satisfying the 2nd condition}}$$

$$= \frac{P(\vec{u}_1, \vec{u}_2) d\vec{u}_1 d\vec{u}_2}{P(\vec{u}_2) d\vec{u}_2}$$

$$P(\vec{u}_1 \mid \vec{u}_2) = \frac{P(\vec{u}_1, \vec{u}_2)}{P(\vec{u}_2)}$$

If two conditions are totally independent, i.e.

$$P(\vec{u}_1 \mid \vec{u}_2) d\vec{u}_1 = P(\vec{u}_1) \vec{u}_1$$

then $P(\vec{u}_1, \vec{u}_2) = P(\vec{u}_1) P(\vec{u}_2)$

Example: $u_1 = u(\vec{x}, t_1)$ and $u_2 = u(\vec{x}, t_2)$

(expect to be independent as $t_1 - t_2$ is sufficiently large)

$$\rho(u_1, u_2) = \rho(u(\vec{x}, t_1), u(\vec{x}, t_2)) \equiv \rho(t_1, t_2)$$

$\rho(t_1, t_2) = 1$ if $t_2 = t_1$ (completely dependent)

$\rho(t_1, t_2) = 0$ if $|t_2 - t_1| \to \infty$ (independent)

$|\rho(t_1, t_2)| < 1$ if $|t_2 - t_1| < \Theta$ (dependent to certain extent)

Θ = the order of magnitude of the temporal separation required for significant decorrelation

= the order of L/q (time scale of large eddies)

$$= \int_0^\infty \rho(t_1, t_2) d(t_1 - t_2)$$

Correlations:

$$R(u_1, u_2) \equiv \langle (u_1 - \langle u_1 \rangle)(u_2 - \langle u_2 \rangle) \rangle = \langle u_1' u_2' \rangle$$

Correlation coefficients:

$$\rho(u_1, u_2) \equiv \frac{R(u_1, u_2)}{\sigma_1 \sigma_2} = \frac{\langle u_1' u_2' \rangle}{\sqrt{\langle u_1'^2 \rangle} \sqrt{\langle u_2'^2 \rangle}}$$

(*i*) $-1 \le \rho(u_1, u_2) \le 1$

(*ii*) $\rho(u_1, u_2) = \pm 1$ iff $\sigma_2 u_1 = \pm \sigma_1 u_2$ (deterministically related)

(*ii*) $\rho(u_1, u_2) = 0$ if (not only if) u_1 and u_2 independent

~ Values of correlation coefficients can indicate the extent of mutual dependence between two random variables.

Gauss (normal) Random Variable:

$$P(u) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(u - \langle u \rangle)^2}{2\sigma^2} \right)$$

- ~ determined by its mean and variance completely
- ~ Skewness S = 0
- ~ Kurtosis K = 3
- ~ moments of odd order are all zero
- ~ moments of even order can be related to the second moment

Turbulence velocity is nearly Gaussian

but not its derivatives ($S \neq 0$ and $K \neq 3$).

Central limit theorem:

Suppose $x_1, x_2, \ldots, x_n$ are independent identical random variables with mean μ and variance σ^2.

Define $s \equiv \sum_{i=1}^{n} x_i$ and $\bar{s} \equiv (s - n\mu)/\sqrt{n}\sigma$

Then $P(\bar{s}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\bar{s}^2}{2}\right)$ as $n \to \infty$

That is s is Gaussian with mean nμ and variance $n\sigma^2$.

4. Statistical Tools and Spectral Analysis

Two random variables u_1 and u_2 are Gaussian if

$$P(u_1,u_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{1-\rho^2}\left(\frac{u_1'^2}{2\sigma_1^2} + \frac{u_2'^2}{2\sigma_2^2} - \rho\frac{u_1'u_2'}{\sigma_1\sigma_2}\right)\right)$$

where $u' = u - \langle u \rangle$

$\rho = \langle u_1' u_2' \rangle / \sigma_1\sigma_2$

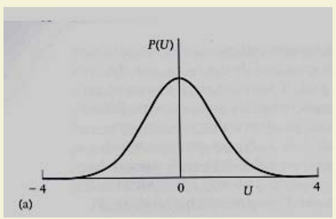
Example: turbulence velocities $u(\vec{x},t)$

$u_1 = u(\vec{x},t)$ and $u_2 = u(\vec{x}+\vec{r},t)$

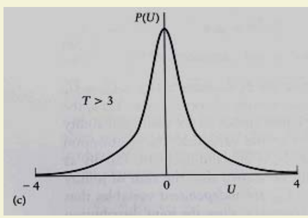
$P(u_1,u_2)$ is nearly joint Gaussian when $|\vec{r}|$ is large.

$P(u_1,u_2)$ is not joint Gaussian when $|\vec{r}|$ is small.

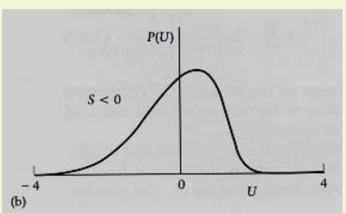
4. Statistical Tools and Spectral Analysis



Gaussian



extensive tails



skewed

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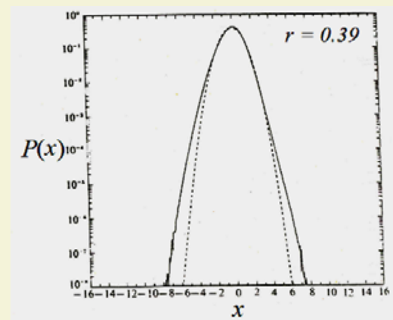
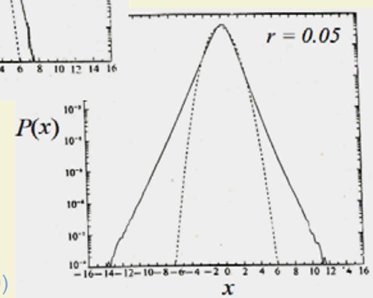


Re_λ = 150



Vincent & Meneguzzi (1990)

Statistical Symmetries

- steadiness (stationary)

~ statistical properties do not change with time

$$u_1 = u(\vec{x}, t_1) \text{ and } u_2 = u(\vec{x}, t_2)$$

$$< u_1 u_2 >= R(t_1, t_2, \vec{x})$$

$$\text{steady} \Rightarrow R(t_1, t_2, \vec{x}) = R(t_1 - t_2, \vec{x})$$

- homogeneity

~ statistical properties are the same at all spatial positions

$$u_1 = u(\vec{x}_1, t) \text{ and } u_2 = u(\vec{x}_2, t)$$

$$< u_1 u_2 >= R(\vec{x}_1, \vec{x}_2, t)$$

$$\text{homogeneous} \Rightarrow R(\vec{x}_1, \vec{x}_2, t) = R(\vec{x}_1 - \vec{x}_2, t)$$

~ infinite flow domain with no boundaries

~ e.g. grid turbulence, small-scale turbulence

~ maybe homogeneous in one or two spatial directions

- isotropy

~ no preferred direction

~ statistical properties remain unchanged as the coordinate system rotates by an arbitrary amount about an arbitrary line, or reflect the flow in any plane

~ rotatonally (spherically) symmetric and statistically invariant under reflection

$$u_1 = u(\vec{x}_1, t) \text{ and } u_2 = u(\vec{x}_2, t)$$

$$< u_1 u_2 >= R(\vec{x}_1, \vec{x}_2, t)$$

$$\text{homogeneous} \Rightarrow R(\vec{x}_1, \vec{x}_2, t) = R(\vec{x}_1 - \vec{x}_2, t)$$

$$\text{isotropic} \Rightarrow R(\vec{x}_1 - \vec{x}_2, t) = R(|\vec{x}_1 - \vec{x}_2|, t)$$

Spectral Analysis

~ proper for homogeneous turbulence

~ convenient for an understanding of scales in turbulence

$$\text{turbulent velocity}: \ \vec{u} = <\vec{u}> + \vec{u}'$$

For simplification, assume for the time being that $<\vec{u}>= 0$

$$u(\vec{x}, t) = \int \int_{-\infty}^{\infty} \int \hat{u}(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} d\vec{k}$$

wave composition

$\text{wave vector} = \vec{k}$

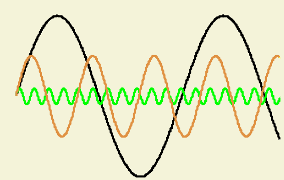
$\text{wave number} = |\vec{k}| = k$

$\text{wave length} = \lambda = 2\pi/k$

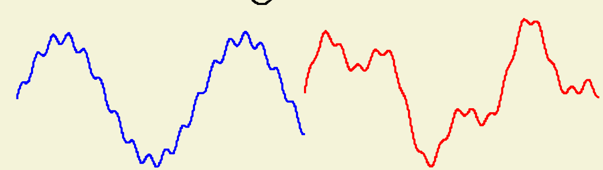
$\text{amplitude} = \hat{u}(\vec{k}, t) d\vec{k}$

Ref. “Fourier Analysis,” T.W. Korner, Cambridge

Example:



$$\begin{cases} \sin(x) \\ 0.5\sin(2.7x) \\ 0.1\sin(11.3x) \end{cases}$$



$\sin(x) + 0.1\sin(11.3x)$ $\sin(x) + 0.5\sin(2.7x) + 0.1\sin(11.3x)$

$$u(\vec{x}, t) = \int \int_{-\infty}^{\infty} \int \hat{u}(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}} d\vec{k}$$

Spectral Analysis

$$u(\vec{x},t)=\int\int\int_{-\infty}^{\infty}\hat{u}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}d\vec{k}$$

$$\hat{u}(\vec{k},t)=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}u(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}d\vec{x}$$

$$\hat{u}(-\vec{k},t)=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}u(\vec{x},t)e^{i\vec{k}\cdot\vec{x}}d\vec{x}$$

$$\hat{u}^*(\vec{k},t)=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}u^*(\vec{x},t)e^{i\vec{k}\cdot\vec{x}}d\vec{x}$$

If u is real, then $\hat{u}(-\vec{k},t)=\hat{u}^*(\vec{k},t)$

Energy spectrum $E(k)$

$E(k)dk$ = kinetic energy per unit mass contained in the Fourier modes having wave number $\in(k,k+dk)$

$$\text{kinetic energy}=\tfrac{1}{2}\iiint u_i(\vec{x})u_i(\vec{x})d\vec{x}$$

$$=\tfrac{1}{2}\iiint\underbrace{\iiint\hat{u}_i(\vec{k})e^{-i\vec{k}\cdot\vec{x}}d\vec{k}}\cdot\underbrace{\iiint\hat{u}_i(\vec{k}')e^{-i\vec{k}'\cdot\vec{x}}d\vec{k}'}d\vec{x}$$

$$=\tfrac{1}{2}\iiint d\vec{k}\iiint d\vec{k}'\ \hat{u}_i(\vec{k})\hat{u}_i(\vec{k}')\iiint e^{-i(\vec{k}+\vec{k}')\vec{x}}d\vec{x}$$

$$=\tfrac{1}{2}(2\pi)^3\iiint d\vec{k}\iiint d\vec{k}'\ \hat{u}_i(\vec{k})\hat{u}_i(\vec{k}')\delta(\vec{k}+\vec{k}')$$

$$=\tfrac{1}{2}(2\pi)^3\iiint\hat{u}_i(\vec{k})\hat{u}_i(-\vec{k})\,d\vec{k}$$

$$=\tfrac{1}{2}(2\pi)^3\iiint\hat{u}_i(\vec{k})\hat{u}_i^*(\vec{k})d\vec{k}$$

$$u(\vec{x},t)=\int\int\int_{-\infty}^{\infty}\hat{u}(\vec{k},t)e^{i\vec{k}\cdot\vec{x}}d\vec{k}\qquad \hat{u}(\vec{k},t)=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}u(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}d\vec{x}$$

e.g. Dirac function

$$\delta(\vec{k})\equiv\begin{cases}\infty & \text{if } \vec{k}=\vec{0}\\ 0 & \text{otherwise}\end{cases}=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}e^{i\vec{k}\cdot\vec{x}}d\vec{x}$$

<pf> Let $u(\vec{x})=1=1\cdot e^{i\vec{0}\cdot\vec{x}}=\dfrac{1}{d\vec{k}}\cdot e^{i\vec{0}\cdot\vec{x}}d\vec{k}$

Obviously,

$$\hat{u}(\vec{k})=\begin{cases}1/d\vec{k}\to\infty & \text{for } \vec{k}=\vec{0}\\ 0 & \text{otherwise}\end{cases}=\delta(\vec{k})$$

$$\Rightarrow\ \delta(\vec{k})=\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{\infty}e^{-i\vec{k}\cdot\vec{x}}d\vec{x}\quad\text{and}\quad\int_{-\infty}^{\infty}\delta(\vec{k})d\vec{k}=1$$