Dynamics of a twelve-car train model

We derive a twelve-car train model, where the dynamic equations of each car are similar to Appendix A except: (1) the first car has a parallel spring/damper set at the rear; (2) the middle cars have both front and rear connections; (3) the last car has a parallel spring/damper set at the front. The three cases are considered as in the following.

(1) Dynamics of the first car:

The dynamics of the first car is similar to Eqs. (3)–(60), except the car-body Eqs. (3)–(12). Therefore, we use the same parameters of Table Ⅱ and use y_c^F, z_c^F, ϕ_c^F, θ_c^F, and ψ_c^F to represent the lateral, vertical, roll, pitch, and yaw motions of the car-body, and y_{t1}^F, z_{t1}^F, ϕ_{t1}^F, ψ_{t1}^F (y_{t2}^F, z_{t2}^F, ϕ_{t2}^F, ψ_{t2}^F) to represent the lateral, vertical, roll, and yaw motions of the front (rear) bogie of the *Fth*-car (F=1 for the first car). As shown in Fig. 5, the connection is modeled as an equivalent spring with K_{by}=20kN/m located at h_k=0.75m and an equivalent lateral damper with C_{by}=50kNs/m located at h_c=1.5m above the center of gravity of the car body. In addition, we assume the distance between the centers of gravity of each car is L_{con}=10.3m. The following derived equations are used to replace Eqs. (3)–(12) for the first car, where the boldface represent the extra terms caused by the connected spring-damper set:

$$m_c \ddot{y}_c^F = F_{syc}^F \tag{61}$$

$$m_c \ddot{z}_c^F = F_{szc}^F \tag{62}$$

$$I_{cx} \ddot{\phi}_c^F = M_{sxc}^F \tag{63}$$

$$I_{cy} \ddot{\theta}_c^F = M_{syc}^F \tag{64}$$

$$I_{cz} \ddot{\psi}_c^F = M_{szc}^F \tag{65}$$

in which the suspension forces and moments acting on the *F*-car of the car body are derived as follows:

$$\begin{aligned} F_{syc}^F = {} & 2K_{sy} y_{t1}^F + 2C_{sy} \dot{y}_{t1}^F + 2K_{sy} y_{t2}^F + 2C_{sy} \dot{y}_{t2}^F - 4K_{sy} y_c^F \\ & - 4C_{sy} \dot{y}_c^F - 4K_{sy} h_S \phi_c^F - 4C_{sy} h_S \dot{\phi}_c^F \\ & \boldsymbol{- C_{by} (\dot{y}_c^F - \dot{y}_c^{F+1}) - K_{by} (y_c^F - y_c^{F+1})} \\ & \boldsymbol{- C_{by} h_S (\dot{\phi}_c^F - \dot{\phi}_c^{F+1}) + C_{by} h_c (\dot{\phi}_c^F - \dot{\phi}_c^{F+1})} \\ & \boldsymbol{- C_{by} L_{con} (\dot{\psi}_c^F + \dot{\psi}_c^{F+1}) - K_{by} L_{con} (\psi_c^F + \psi_c^{F+1})} \\ & \boldsymbol{- K_{by} h_S (\phi_c^F - \phi_c^{F+1}) + K_{by} h_k (\phi_c^F - \phi_c^{F+1})} \end{aligned} \tag{66}$$

$$F_{szc}^F = 2K_{sz} z_{t1}^F + 2C_{sz} \dot{z}_{t1}^F + 2K_{sz} z_{t2}^F + 2C_{sz} \dot{z}_{t2}^F - 4K_{sz} z_c^F - 4C_{sz} \dot{z}_c^F \tag{67}$$

$$\begin{aligned} M_{sxc}^F = {} & 2K_{sy} h_S (y_{t1}^F + y_{t2}^F) + 2C_{sy} h_S (\dot{y}_{t1}^F + \dot{y}_{t2}^F) \\ & + 2b_2^2 K_{sz} (\phi_{t1}^F + \phi_{t2}^F) + 2b_2^2 C_{sz} (\dot{\phi}_{t1}^F + \dot{\phi}_{t2}^F) \\ & - 4K_{sz} b_2^2 \phi_c^F - 4C_{sz} b_2^2 \dot{\phi}_c^F - 4K_{sy} h_S^2 \phi_c^F \\ & - 4C_{sy} h_S^2 \dot{\phi}_c^F - 4h_S (K_{sy} y_c^F + C_{sy} \dot{y}_c^F) \\ & \boldsymbol{+ C_{by} (h_S - h_c) \dot{y}_c^{F+1} - C_{by} (h_S - h_c) \dot{y}_c^F} \\ & \boldsymbol{- K_{by} (h_S - h_k) y_c^F + K_{by} (h_S - h_k) y_c^{F+1}} \\ & \boldsymbol{- C_{by} (h_S - h_c)^2 \dot{\phi}_c^F + C_{by} (h_S - h_c)^2 \dot{\phi}_c^{F+1}} \\ & \boldsymbol{- K_{by} (h_S - h_k)^2 \phi_c^F + K_{by} (h_S - h_k)^2 \phi_c^{F+1}} \\ & \boldsymbol{- C_{by} L_{con} (h_S - h_c) \dot{\psi}_c^F - K_{by} L_{con} (h_S - h_k) \psi_c^F} \\ & \boldsymbol{- C_{by} L_{con} (h_S - h_c) \dot{\psi}_c^{F+1} - K_{by} L_{con} (h_S - h_k) \psi_c^{F+1}} \end{aligned} \tag{68}$$

$$\begin{aligned} M_{syc}^F = {} & 2K_{sz} L_2 (-z_{t1}^F + z_{t2}^F) + 2C_{sz} L_2 (-\dot{z}_{t1}^F + \dot{z}_{t2}^F) \\ & - (4K_{sz} L_2^2 + 4K_{sx} h_S^2) \theta_c^F - (4C_{sz} L_2^2 + 4C_{sx} h_S^2) \dot{\theta}_c^F \end{aligned} \tag{69}$$

$$
\begin{aligned}
M_{szc}^{F} = {} & 2K_{sy}L_{2}(y_{t1}^{F}-y_{t2}^{F})+2C_{sy}L_{2}(\dot{y}_{t1}^{F}-\dot{y}_{t2}^{F}) \\
& +2K_{sx}b_{2}^{2}(\psi_{t1}^{F}+\psi_{t2}^{F})+2C_{sx}b_{2}^{2}(\dot{\psi}_{t1}^{F}+\dot{\psi}_{t2}^{F}) \\
& -4L_{2}^{2}(K_{sy}\psi_{c}^{F}+C_{sy}\dot{\psi}_{c}^{F})-4b_{2}^{2}(K_{sx}\psi_{c}^{F}+C_{sx}\dot{\psi}_{c}^{F}) \\
& \boldsymbol{-C_{by}L_{con}(\dot{y}_{c}^{F}-\dot{y}_{c}^{F+1})-K_{by}L_{con}(y_{c}^{F}-y_{c}^{F+1})} \\
& \boldsymbol{-C_{by}L_{con}h_{S}(\dot{\phi}_{c}^{F}-\dot{\phi}_{c}^{F+1})+C_{by}L_{con}h_{c}(\dot{\phi}_{c}^{F}-\dot{\phi}_{c}^{F+1})} \\
& \boldsymbol{-C_{by}L_{con}^{2}(\dot{\psi}_{c}^{F}+\dot{\psi}_{c}^{F+1})-K_{by}L_{con}h_{S}(\phi_{c}^{F}-\phi_{c}^{F+1})} \\
& \boldsymbol{+K_{by}L_{con}h_{k}(\phi_{c}^{F}-\phi_{c}^{F+1})-K_{by}L_{con}^{2}(\psi_{c}^{F}-\psi_{c}^{F+1})} \\
& \boldsymbol{-2K_{by}L_{con}^{2}\psi_{c}^{F+1}}
\end{aligned}
\tag{70}
$$

(2) Dynamics of the middle cars:

The dynamics of the middle cars is derived in a similar way. We use y_c^M, z_c^M, ϕ_c^M, θ_c^M, and ψ_c^M to represent the lateral, vertical, roll, pitch, and yaw motions of the car-body, and y_{t1}^M, z_{t1}^M, ϕ_{t1}^M, ψ_{t1}^M (y_{t2}^M, z_{t2}^M, ϕ_{t2}^M, ψ_{t2}^M) to represent the lateral, vertical, roll, and yaw motions of the front (rear) bogie of the *Mth*-car (M=2~11 for the middle cars). The following derived equations are used to replace Eqs. (3)–(12) for the middle cars, where the boldface represent the extra terms caused by the connected spring-damper sets:

$$m_c\ddot{y}_c^M = F_{syc}^M \tag{71}$$

$$m_c\ddot{z}_c^M = F_{szc}^M \tag{72}$$

$$I_{cx}\ddot{\phi}_c^M = M_{sxc}^M \tag{73}$$

$$I_{cy}\ddot{\theta}_c^M = M_{syc}^M \tag{74}$$

$$I_{cz}\ddot{\psi}_c^M = M_{szc}^M \tag{75}$$

in which the suspension forces and moments acting on the *M*-car of the car body are derived as follows:

$$
\begin{aligned}
F_{syc}^{M} &= 2K_{sy}y_{t1}^{M} + 2C_{sy}\dot{y}_{t1}^{M} + 2K_{sy}y_{t2}^{M} + 2C_{sy}\dot{y}_{t2}^{M} - 4K_{sy}y_{c}^{M} \\
&\quad - 4C_{sy}\dot{y}_{c}^{M} - 4K_{sy}h_{S}\phi_{c}^{M} - 4C_{sy}h_{S}\dot{\phi}_{c}^{M} \\
&\quad + C_{by}(\dot{y}_{c}^{M-1} - 2\dot{y}_{c}^{M} + \dot{y}_{c}^{M+1}) + K_{by}(y_{c}^{M-1} - 2y_{c}^{M} + y_{c}^{M+1}) \\
&\quad + C_{by}h_{S}(\dot{\phi}_{c}^{M-1} - 2\dot{\phi}_{c}^{M} + \dot{\phi}_{c}^{M+1}) - C_{by}h_{c}(\dot{\phi}_{c}^{M-1} - 2\dot{\phi}_{c}^{M} + \dot{\phi}_{c}^{M+1}) \\
&\quad + C_{by}L_{con}(\dot{\psi}_{c}^{M-1} - \dot{\psi}_{c}^{M+1}) + K_{by}L_{con}(\psi_{c}^{M-1} - \psi_{c}^{M+1}) \\
&\quad + K_{by}h_{S}(\phi_{c}^{M-1} - 2\phi_{c}^{M} + \phi_{c}^{M+1}) - K_{by}h_{k}(\phi_{c}^{M-1} - 2\phi_{c}^{M} + \phi_{c}^{M+1})
\end{aligned} \tag{76}
$$

$$
F_{szc}^{M} = 2K_{sz}z_{t1}^{M} + 2C_{sz}\dot{z}_{t1}^{M} + 2K_{sz}z_{t2}^{M} + 2C_{sz}\dot{z}_{t2}^{M} - 4K_{sz}z_{c}^{M} - 4C_{sz}\dot{z}_{c}^{M} \tag{77}
$$

$$
\begin{aligned}
M_{sxc}^{M} &= 2K_{sy}h_{S}(y_{t1}^{M} + y_{t2}^{M}) + 2C_{sy}h_{S}(\dot{y}_{t1}^{M} + \dot{y}_{t2}^{M}) + 2b_{2}^{2}K_{sz}(\phi_{t1}^{M} + \phi_{t2}^{M}) \\
&\quad + 2b_{2}^{2}C_{sz}(\dot{\phi}_{t1}^{M} + \phi_{t2}^{M}) - 4K_{sz}b_{2}^{2}\phi_{c}^{M} - 4C_{sz}b_{2}^{2}\dot{\phi}_{c}^{M} - 4K_{sy}h_{S}^{2}\phi_{c}^{M} \\
&\quad - 4C_{sy}h_{S}^{2}\dot{\phi}_{c}^{M} - 4h_{S}(K_{sy}y_{c}^{M} + C_{sy}\dot{y}_{c}^{M}) \\
&\quad + C_{by}(h_{S} - h_{c})\dot{y}_{c}^{M-1} - 2C_{by}(h_{S} - h_{c})\dot{y}_{c}^{M} + C_{by}(h_{S} - h_{c})\dot{y}_{c}^{M+1} \\
&\quad + K_{by}(h_{S} - h_{k})y_{c}^{M-1} - 2K_{by}(h_{S} - h_{k})y_{c}^{M} + K_{by}(h_{S} - h_{k})y_{c}^{M+1} \\
&\quad + C_{by}(h_{S} - h_{c})^{2}\dot{\phi}_{c}^{M-1} - 2C_{by}(h_{S} - h_{c})^{2}\dot{\phi}_{c}^{M} + C_{by}(h_{S} - h_{c})^{2}\dot{\phi}_{c}^{M+1} \\
&\quad + K_{by}(h_{S} - h_{k})^{2}\phi_{c}^{M-1} - 2K_{by}(h_{S} - h_{k})^{2}\phi_{c}^{M} + K_{by}(h_{S} - h_{k})^{2}\phi_{c}^{M+1} \\
&\quad + C_{by}L_{con}(h_{S} - h_{c})\dot{\psi}_{c}^{M-1} + K_{by}L_{con}(h_{S} - h_{k})\psi_{c}^{M-1} \\
&\quad - C_{by}L_{con}(h_{S} - h_{c})\dot{\psi}_{c}^{M+1} - K_{by}L_{con}(h_{S} - h_{k})\psi_{c}^{M+1}
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
M_{syc}^{M} &= 2K_{sz}L_{2}(-z_{t1}^{M} + z_{t2}^{M}) + 2C_{sz}L_{2}(-\dot{z}_{t1}^{M} + \dot{z}_{t2}^{M}) \\
&\quad - (4K_{sz}L_{2}^{2} + 4K_{sx}h_{S}^{2})\theta_{c}^{M} - (4C_{sz}L_{2}^{2} + 4C_{sx}h_{S}^{2})\dot{\theta}_{c}^{M}
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
M_{szc}^{M} &= 2K_{sy}L_{2}(y_{t1}^{M} - y_{t2}^{M}) + 2C_{sy}L_{2}(\dot{y}_{t1}^{M} - \dot{y}_{t2}^{M}) \\
&\quad + 2C_{sx}b_{2}^{2}(\psi_{t1}^{M} + \psi_{t2}^{M}) + 2K_{sx}b_{2}^{2}(\psi_{t1}^{M} + \psi_{t2}^{M}) \\
&\quad - 4L_{2}^{2}(K_{sy}\psi_{c}^{M} + C_{sy}\dot{\psi}_{c}^{M}) - 4b_{2}^{2}(K_{sx}\psi_{c}^{M} + C_{sx}\dot{\psi}_{c}^{M}) \\
&\quad - C_{by}L_{con}(y_{c}^{M-1} - y_{c}^{M+1}) - K_{by}L_{con}(y_{c}^{M-1} - y_{c}^{M+1}) \\
&\quad - C_{by}L_{con}h_{S}(\phi_{c}^{M-1} - \phi_{c}^{M+1}) + C_{by}L_{con}h_{c}(\phi_{c}^{M-1} - \phi_{c}^{M+1}) \\
&\quad - C_{by}L_{con}^{2}(\dot{\psi}_{c}^{M-1} + 2\dot{\psi}_{c}^{M} + \dot{\psi}_{c}^{M+1}) - K_{by}L_{con}^{2}(\psi_{c}^{M-1} + 2\psi_{c}^{M} + \psi_{c}^{M+1}) \\
&\quad - K_{by}L_{con}h_{S}(\phi_{c}^{M-1} - \phi_{c}^{M+1}) + K_{by}L_{con}h_{k}(\phi_{c}^{M-1} - \phi_{c}^{M+1})
\end{aligned} \tag{80}
$$

(3) Dynamics of the last car:

Similarly, we use the same parameters of Table Ⅱ and use y_c^L, z_c^L, ϕ_c^L, θ_c^L, and ψ_c^L to represent the lateral, vertical, roll, pitch, and yaw motions of the car-body, and y_{t1}^L, z_{t1}^L, ϕ_{t1}^L, ψ_{t1}^L (y_{t2}^L, z_{t2}^L, ϕ_{t2}^L, ψ_{t2}^L) to represent the lateral, vertical, roll, and yaw motions of the front (rear) bogie of the *Lth*-car (*L*=12 for the last car). The following derived equations are used to replace Eqs. (3)–(12) for the last car, where the boldface represent the extra terms caused by the connected spring-damper set:

$$m_c \ddot{y}_c^L = F_{syc}^L \tag{81}$$

$$m_c \ddot{z}_c^L = F_{szc}^L \tag{82}$$

$$I_{cx} \ddot{\phi}_c^L = M_{sxc}^L \tag{83}$$

$$I_{cy} \ddot{\theta}_c^L = M_{syc}^L \tag{84}$$

$$I_{cz} \ddot{\psi}_c^L = M_{szc}^L \tag{85}$$

in which the suspension forces and moments acting on the *L*-car of the car body are derived as follows:

$$\begin{aligned}
F_{syc}^L = {} & 2K_{sy} y_{t1}^L + 2C_{sy} \dot{y}_{t1}^L + 2K_{sy} y_{t2}^L + 2C_{sy} \dot{y}_{t2}^L - 4K_{sy} y_c^L \\
& - 4C_{sy} \dot{y}_c^L - 4K_{sy} h_S \phi_c^L - 4C_{sy} h_S \dot{\phi}_c^L \\
& \boldsymbol{+ C_{by} (\dot{y}_c^{L-1} - \dot{y}_c^L) + K_{by} (y_c^{L-1} - y_c^L)} \\
& \boldsymbol{+ C_{by} h_S (\dot{\phi}_c^{L-1} - \dot{\phi}_c^L) - C_{by} h_c (\dot{\phi}_c^{L-1} - \dot{\phi}_c^L)} \\
& \boldsymbol{+ C_{by} L_{con} (\dot{\psi}_c^{L-1} + \dot{\psi}_c^L) + K_{by} L_{con} (\psi_c^{L-1} + \psi_c^L)} \\
& \boldsymbol{+ K\, h_S (\phi^{L-1} - \phi^L) - K\, h_c (\phi^{L-1} - \phi^L)}
\end{aligned} \tag{86}$$

$$F_{szc}^{L} = 2K_{sz} z_{t1}^{L} + 2C_{sz} \dot{z}_{t1}^{L} + 2K_{sz} z_{t2}^{L} + 2C_{sz} \dot{z}_{t2}^{L} - 4K_{sz} z_{c}^{L} - 4C_{sz} \dot{z}_{c}^{L} \tag{87}$$

$$\begin{aligned}
M_{sxc}^{L} &= 2K_{sy} h_S (y_{t1}^{L} + y_{t2}^{L}) + 2C_{sy} h_S (\dot{y}_{t1}^{L} + \dot{y}_{t2}^{L}) \\
&+ 2b_2^2 K_{sz} (\phi_{t1}^{L} + \phi_{t2}^{L}) + 2b_2^2 C_{sz} (\dot{\phi}_{t1}^{L} + \dot{\phi}_{t2}^{L}) \\
&- 4K_{sz} b_2^2 \phi_{c}^{L} - 4C_{sz} b_2^2 \dot{\phi}_{c}^{L} - 4K_{sy} h_S^2 \phi_{c}^{L} \\
&- 4C_{sy} h_S^2 \dot{\phi}_{c}^{L} - 4h_S (K_{sy} y_{c}^{L} + C_{sy} \dot{y}_{c}^{L}) \\
&+ \boldsymbol{C_{by}(h_S - h_c) \dot{y}_{c}^{L-1} - C_{by}(h_S - h_c) \dot{y}_{c}^{L}} \\
&+ \boldsymbol{K_{by}(h_S - h_k) y_{c}^{L-1} - K_{by}(h_S - h_k) y_{c}^{L}} \\
&+ \boldsymbol{C_{by}(h_S - h_c)^2 \dot{\phi}_{c}^{L-1} - C_{by}(h_S - h_c)^2 \dot{\phi}_{c}^{L}} \\
&+ \boldsymbol{K_{by}(h_S - h_k)^2 \phi_{c}^{L-1} - K_{by}(h_S - h_k)^2 \phi_{c}^{L}} \\
&+ \boldsymbol{C_{by} L_{con} (h_S - h_c) \dot{\psi}_{c}^{L-1} + K_{by} L_{con} (h_S - h_k) \psi_{c}^{L-1}} \\
&+ \boldsymbol{C_{by} L_{con} (h_S - h_c) \dot{\psi}_{c}^{L} + K_{by} L_{con} (h_S - h_k) \psi_{c}^{L}}
\end{aligned} \tag{88}$$

$$\begin{aligned}
M_{syc}^{L} &= 2K_{sz} L_2 (-z_{t1}^{L} + z_{t2}^{L}) + 2C_{sz} L_2 (-\dot{z}_{t1}^{L} + \dot{z}_{t2}^{L}) \\
&- (4K_{sz} L_2^2 + 4K_{sx} h_S^2) \theta_{c}^{L} - (4C_{sz} L_2^2 + 4C_{sx} h_S^2) \dot{\theta}_{c}^{L}
\end{aligned} \tag{89}$$

$$\begin{aligned}
M_{szc}^{L} &= 2K_{sy} L_2 (y_{t1}^{L} - y_{t2}^{L}) + 2C_{sy} L_2 (\dot{y}_{t1}^{L} - \dot{y}_{t2}^{L}) \\
&+ 2K_{sx} b_2^2 (\psi_{t1}^{L} + \psi_{t2}^{L}) + 2C_{sx} b_2^2 (\dot{\psi}_{t1}^{L} + \dot{\psi}_{t2}^{L}) \\
&- 4L_2^2 (K_{sy} \psi_{c}^{L} + C_{sy} \dot{\psi}_{c}^{L}) - 4b_2^2 (K_{sx} \psi_{c}^{L} + C_{sx} \dot{\psi}_{c}^{L}) \\
&\boldsymbol{- C_{by} L_{con} (\dot{y}_{c}^{L-1} - \dot{y}_{c}^{L}) - K_{by} L_{con} (y_{c}^{L-1} - y_{c}^{L})} \\
&\boldsymbol{- C_{by} L_{con} h_S (\dot{\phi}_{c}^{L-1} - \dot{\phi}_{c}^{L}) + C_{by} L_{con} h_c (\dot{\phi}_{c}^{L-1} - \dot{\phi}_{c}^{L})} \\
&\boldsymbol{- C_{by} L_{con}^2 (\dot{\psi}_{c}^{L-1} + \dot{\psi}_{c}^{L}) - K_{by} L_{con} h_S (\phi_{c}^{L-1} - \phi_{c}^{L})} \\
&\boldsymbol{+ K_{by} L_{con} h_k (\phi_{c}^{L-1} - \phi_{c}^{L}) - K_{by} L_{con}^2 (\psi_{c}^{L-1} - \psi_{c}^{L})} \\
&\boldsymbol{- 2K_{by} L_{con}^2 \psi_{c}^{L}}
\end{aligned} \tag{90}$$

The dynamic model of the twelve-car DOF train model can be derived by Eqs. (13)–(90).